Order Independence and Rationalizability

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Abstract

Two natural strategy elimination procedures have been studied for strategic games. The first one involves the notion of (strict, weak, etc) dominance and the second the notion of rationalizability. In the case of dominance the criterion of order independence allowed us to clarify which notions and under what circumstances are robust. In the case of rationalizability this criterion has not been considered.

In this paper we investigate the problem of order independence for rationalizability by focusing on three naturally entailed reduction relations on games. These reduction relations are distinguished by the adopted reference point for the notion of a better response. Additionally, they are parametrized by the adopted system of beliefs.

We show that for one reduction relation the outcome of its (possibly transfinite) iterations does not depend on the order of elimination of the strategies. This result does not hold for the other two reduction relations. However, under a natural assumption the iterations of all three reduction relations yield the same outcome.

The obtained order independence results apply to the frameworks considered in Bernheim [1984] and Pearce [1984]. For finite games the iterations of all three reduction relations coincide and the order independence holds for three natural systems of beliefs considered in the literature.

1 Introduction

Rationalizability was introduced in Bernheim [1984] and Pearce [1984] to formalize the intuition that players in non-cooperatives games act by having common knowledge of each others' rational behaviour. Rationalizable strategies in a strategic game are defined as a limit of an iterative process in which one repeatedly removes the strategies that are never best responses (NBR) to the beliefs held about the other players. In contrast to the iterated elimination of strictly and of weakly dominated strategies at each stage all 'undesirable' strategies are removed.

Much attention was devoted in the literature to the issue of order independence for the iterated elimination of strictly and of weakly dominated strategies. It is well-known that strict dominance is order independent for finite games (see Gilboa, Kalai and Zemel [1990] and Stegeman [1990]), while weak dominance is order dependent. This has been often used as an argument in support of the first procedure and against the second one, see, e.g., Osborne and Rubinstein [1994]. On the other hand, Dufwenberg and Stegeman [2002] indicated that order independence for strict dominance fails for arbitrary games though does hold for a large class of infinite games.

The criterion of order independence did not seem to be applied to assess the merits of the iterated elimination of NBR. In this paper we study this problem by analyzing what happens when

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at each stage of the iterative process only some strategies that are NBR are eliminated. This brings us to a study of three naturally entailed reduction relations. They are distinguished by the adopted reference point for the notion of a better response, which can be the initial game, the game currently being reduced or the reduced game. Additionally, they are parametrized by the adopted system of beliefs. In general these relations differ and transfinite iterations are possible.

We show for one reduction relation that for all 'well-behaving' systems of beliefs the outcome of the iterated elimination of strategies does not depend on the order of elimination. The result does not hold for the other two reduction relations, even for two-person games and beliefs being the strategies of the opponent.

Further, using a game modeling a version of Bertrand competition between two firms we show that the variants of these reduction relations in which all strategies that are NBR are eliminated differ, as well. The same example also shows that the relation considered in Bernheim [1984], according to which at each stage *all* strategies that are NBR are eliminated, yields a weaker reduction than the one according to which at each stage only *some* strategies that are NBR are eliminated. In other words, natural games exist in which it is beneficial to eliminate at certain stages only some strategies that are NBR.

The situation changes if we assume that for each belief μ_i in a restriction G of the original game a best response to μ_i in G exists. We show that then the iterations of all three reduction relations yield the same outcome. This implies order independence for all three reduction relations for the class of games for which Dufwenberg and Stegeman [2002] established order independence of the iterated elimination of strictly dominated strategies.

A complicating factor in these considerations is that iterations of each of the reduction relation can reduce the initial game to an empty game. We discuss natural examples of games for which the unique outcome of the iterated elimination process is a non-empty game. In particular, order independence and non-emptiness of the final outcome holds for a relaxation of two elimination procedures studied in the literature:

- the one considered in Bernheim [1984], concerning a compact game with continuous payoff functions, in which at each stage we now eliminate only *some* strategies that are NBR (to the joint strategies of the opponents), and
- the one considered in Pearce [1984], concerning mixed extension of a finite game, in which at each stage we now eliminate only *some* mixed strategies that are NBR (to the elements of the products of convex hulls of the opponents' strategies).

The definition of rationalizable strategies is parameterized by a system of belief. In the case of finite games three natural alternatives were considered:

- joint pure strategies of the opponents, see, e.g., Bernheim [1984],
- joint mixed strategies of the opponents, see, e.g., Bernheim [1984] and Pearce [1984],
- probability distributions over the joint pure strategies of the opponents, see, e.g., Bernheim [1984] and Osborne and Rubinstein [1994].

A direct consequence of our results is that for finite games order independence holds for all three reduction relations and all three alternatives of the systems of belief.

In summary, all three versions of the iterated elimination of NBR are order independent for the same classes of games for which iterated elimination of strictly dominated strategies was established. Additionally, for one version order independence holds for all 'well-behaving' systems of beliefs.

2 Preliminaries

Given n players we represent a strategic game (in short, a game) by a sequence

$$(S_1, \ldots, S_n, p_1, \ldots, p_n),$$

where for each $i \in [1..n]$

- S_i is the non-empty set of ***strategies*** available to player i,
- p_i is the payoff function for the player i, so $p_i : S_1 \times \ldots \times S_n \rightarrow \mathcal{R}$, where $\mathcal{R}$ is the set of real numbers.

Given a sequence of non-empty sets of strategies $S_1, \ldots, S_n$ and $s \in S_1 \times \ldots \times S_n$ we denote the ith element of s by s_i and use the following standard notation:

- $s_{-i} := (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$,
- $(s'_i, s_{-i}) := (s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)$, where we assume that $s'_i \in S_i$,
- $S_{-i} := S_1 \times \ldots \times S_{i-1} \times S_{i+1} \times \ldots \times S_n$.

We denote the strategies of player i by s_i, possibly with some superscripts.

By a ***restriction*** of a game $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$ we mean a game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ such that each S_i is a (possibly empty) subset of T_i and each p_i is identified with its restriction to the smaller domain. We write then $G \subseteq H$.

If some S_i is empty, we call G a ***degenerate restriction*** of H. In this case the references to $p_j(s)$ (for any $j \in [1..n]$) are incorrect and we shall need to be careful about this. If all S_i are empty, we call G an ***empty game*** and denote it by $\emptyset_n$. If no S_i is empty, we call G a ***non-degenerate restriction*** of H.

Similarly, we introduce the notions of a ***union*** and ***intersection*** of a transfinite sequence $(G_\alpha)_{\alpha<\gamma}$ of restrictions of H (α and γ are ordinals) denoted respectively by $\bigcup_{\alpha<\gamma} G_\alpha$ and $\bigcap_{\alpha<\gamma} G_\alpha$.

3 Belief structures

We assume that each player i in the game $H = (T_1, \ldots, T_n, p_1, \ldots, p_n)$ has some further unspecified non-empty set of beliefs $\mathcal{B}_i$ about his opponents. We call then $\mathcal{B} := (\mathcal{B}_1, \ldots, \mathcal{B}_n)$, a ***belief system*** in the game H. We further assume that each payoff function p_i can be modified to an ***expected payoff*** function $p_i : S_i \times \mathcal{B}_i \rightarrow \mathcal{R}$.

Then we say that a strategy s_i of player i is a ***best response*** to a belief $\mu_i \in \mathcal{B}_i$ in H if for all strategies $s'_i \in T_i$

$$p_i(s_i, \mu_i) \geq p_i(s'_i, \mu_i).$$

In what follows we also assume that each set of beliefs $\mathcal{B}_i$ of player i in H can be *narrowed* to any restriction G of H. We denote the outcome of this ***narrowing*** of $\mathcal{B}_i$ to G by $\mathcal{B}_i \dot{\cap} G$. The beliefs in $\mathcal{B}_i \dot{\cap} G$ can be also considered as beliefs in the game G. We call then the pair $(\mathcal{B}, \dot{\cap})$, where $\mathcal{B} := (\mathcal{B}_1, \ldots, \mathcal{B}_n)$, a ***belief structure*** in the game H.

Finally, given a belief structure $(\mathcal{B}, \dot{\cap})$ in a game H we say that a restriction G of H is ***$\mathcal{B}$-closed*** if each strategy s_i of player i in G is a best response in H (note this reference to H and *not* G) to a belief in $\mathcal{B}_i \dot{\cap} G$.

Fix now a game $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$ and a belief structure $(\mathcal{B}, \dot{\cap})$ in H. The following natural property of $\dot{\cap}$ will be relevant.

A If $G_1 \subseteq G_2 \subseteq H$, then for all $i \in [1..n]$, $\mathcal{B}_i \dot{\cap} G_1 \subseteq \mathcal{B}_i \dot{\cap} G_2$.

The following belief structure will be often used. Assume that for each player his set of beliefs $\mathcal{B}_i$ in the game $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$ consists of the joint strategies of the opponents, i.e., $\mathcal{B}_i = T_{-i}$. For a restriction $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ of H we define $T_{-i} \dot{\cap} G := S_{-i}$. Note that property **A** is then satisfied. We call $(\mathcal{B}, \dot{\cap})$ the ***pure*** [1] belief structure in H.

4 Reductions of games

Assume now a game H and a belief structure $(\mathcal{B}, \dot{\cap})$ in H. We introduce a notion of reduction $\leadsto$ between a restriction $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ of H and a restriction $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$ of G defined by:

- $G \leadsto G'$ when $G \neq G'$ and for all $i \in [1..n]$

$$\text{no } s_i \in S_i \setminus S'_i \text{ is a best response in } H \text{ to some } \mu_i \in \mathcal{B}_i \dot{\cap} G.$$

Of course, the $\leadsto$ relation depends on the underlying belief structure $(\mathcal{B}, \dot{\cap})$ in H but we do not indicate this dependence as no confusion will arise. Note that in the definition of $\leadsto$ we do not require that all strategies that are NBR are removed. So in general $G \leadsto G'$ can hold for several restrictions G'. Also, what is important, we refer to the best responses in H and *not* in G or G'. The reduction relations that take these two alternative points of reference will be studied in the next section.

Let us define now appropriate iterations of the $\leadsto$ relation. We shall use this concept for various reduction relations so define it for an arbitrary relation $\longmapsto$ between a restriction G of H and a restriction G' of G.

Definition 4.1 Consider a transfinite sequence of restrictions $(G_\alpha)_{\alpha \leq \gamma}$ of H such that

- $H = G_0$,
- for all $\alpha < \gamma$, $G_\alpha \longmapsto G_{\alpha+1}$,
- for all limit ordinals $\beta \leq \gamma$, $G_\beta = \bigcap_{\alpha < \beta} G_\alpha$,
- for no G', $G_\gamma \longmapsto G'$ holds.

We say then that $(G_\alpha)_{\alpha \leq \gamma}$ is a ***maximal sequence*** of the $\longmapsto$ reductions and call G_γ its ***outcome***. Also, we write $H \longmapsto^\alpha G_\alpha$ for each $\alpha \leq \gamma$. $\Box$

We now establish the following general order independence result.

Theorem 4.2 (Order Independence) *Consider a game H and a belief structure $(\mathcal{B}, \dot{\cap})$ in H. Assume property **A**. Then any maximal sequence of the $\leadsto$ reductions yields the same outcome which is the largest restriction of H that is $\mathcal{B}$-closed.*

Proof. First we establish the following claim.

[1] to indicate that it involves only pure strategies of the opponents

Claim 1 *There exists a largest restriction of H that is $\mathcal{B}$-closed.*

Proof. First note that each empty game is $\mathcal{B}$-closed. Consider now a transfinite sequence of restrictions $(G_\alpha)_{\alpha<\gamma}$ of H such that each G_α is $\mathcal{B}$-closed. We claim that then $\bigcup_{\alpha<\gamma} G_\alpha$ is $\mathcal{B}$-closed, as well.

To see this choose a strategy s_i of player i in $\bigcup_{\alpha<\gamma} G_\alpha$. Then s_i is a strategy of player i in G_{α_0} for some $\alpha_0 < \gamma$. The restriction G_{α_0} is $\mathcal{B}$-closed, so for some $\mu_i \in \mathcal{B}_i \mathrel{\dot\cap} G_{\alpha_0}$ the strategy s_i is a best response to μ_i in H. By property **A** $\mu_i \in \mathcal{B}_i \mathrel{\dot\cap} \bigcup_{\alpha<\gamma} G_\alpha$. □

Consider now a maximal sequence $(G_\alpha)_{\alpha\leq\gamma}$ of the $\rightsquigarrow$ reductions. Take a restriction H' of H such that for some $\alpha < \gamma$

- H' is $\mathcal{B}$-closed,
- $H' \subseteq G_\alpha$.

Consider a strategy s_i of player i in H'. Then s_i is also a strategy of player i in G_α. H' is $\mathcal{B}$-closed, so s_i is a best response in H to a belief $\mu_i \in \mathcal{B}_i \mathrel{\dot\cap} H'$. By property **A** $\mu_i \in \mathcal{B}_i \mathrel{\dot\cap} G_\alpha$. So by the definition of the $\rightsquigarrow$ reduction the strategy s_i is not deleted in the transition $G_\alpha \rightsquigarrow G_{\alpha+1}$, i.e., s_i is a strategy of player i in $G_{\alpha+1}$. Hence $H' \subseteq G_{\alpha+1}$.

We conclude by transfinite induction that $H' \subseteq G_\gamma$. In particular we conclude that $G_{\mathcal{B}} \subseteq G_\gamma$, where $G_{\mathcal{B}}$ is the largest restriction of H that is $\mathcal{B}$-closed and the existence of which is guaranteed by Claim 1.

But also $G_\gamma \subseteq G_{\mathcal{B}}$ since G_γ is $\mathcal{B}$-closed and $G_{\mathcal{B}}$ is the largest restriction of H that is $\mathcal{B}$-closed. □

Since at each stage of the above elimination process some strategy is removed, this iterated elimination process eventually stops, i.e., the considered maximal sequences always exist. The result can be interpreted as a statement that each, possibly transfinite, iterated elimination of NBR yields the same outcome.

The $\rightsquigarrow$ reduction allows us to remove only *some* strategies that are NBR in the initial game H. If we remove *all* strategies that are NBR, we get the reduction relation that corresponds to the ones considered in the literature for specific belief structures. It is defined as follows. Consider a restriction $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ of H and a restriction $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$ of G. We define then the 'fast' reduction $\stackrel{f}{\rightsquigarrow}$ by:

- $G \stackrel{f}{\rightsquigarrow} G'$ when $G \neq G'$ and for all $i \in [1..n]$

$$S'_i = \{s_i \in S_i \mid \exists \mu_i \in \mathcal{B}_i \mathrel{\dot\cap} G \, \forall s'_i \in T_i \, p_i(s'_i, \mu_i) \leq p_i(s_i, \mu_i)\}.$$

Since the $\stackrel{f}{\rightsquigarrow}$ reduction removes all strategies that are never best responses, $G \stackrel{f}{\rightsquigarrow} G'$ and $G \rightsquigarrow G''$ implies $G' \subseteq G''$.

We now show that the iterated application of the $\stackrel{f}{\rightsquigarrow}$ reduction yields a stronger reduction than $\rightsquigarrow$ and that $\stackrel{f}{\rightsquigarrow}$ is indeed 'fast' in the sense that it generates reductions of the original game H faster than the $\rightsquigarrow$ reduction. While this is of course as expected, we shall see in the next section that these properties do not hold for a simple variant of the $\rightsquigarrow$ reduction studied in the literature.

Theorem 4.3 *Consider a game H and a belief structure $(\mathcal{B}, \dot\cap)$ in H. Assume property **A**.*

(i) Suppose $G \stackrel{f}{\rightsquigarrow}{}^{\gamma} G'$ and $G \rightsquigarrow^{\gamma} G''$. Then $G' \subseteq G''$.

(ii) Suppose $H \overset{f}{\rightsquigarrow}^{\beta} G$ *and* $H \rightsquigarrow^{\gamma} G$*. Then* $\beta \leq \gamma$*.*

Proof. First we establish a simple claim concerning the restrictions of H.

Claim 1 *Suppose* $G_1 \subseteq G_2$*,* $G_1 \overset{f}{\rightsquigarrow} G'$ *and* $G_2 \overset{f}{\rightsquigarrow} G''$*. Then* $G' \subseteq G''$*.*

Proof. Let $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$ and $G'' := (S''_1, \ldots, S''_n, p_1, \ldots, p_n)$.

Suppose $s'_i \in S'_i$. Then for some $\mu_i \in \mathcal{B}_i \dot{\cap} G_1$ we have $\forall s^*_i \in T_i \ p_i(s^*_i, \mu_i) \leq p_i(s'_i, \mu_i)$. By property **A** $\mu_i \in \mathcal{B}_i \dot{\cap} G_2$, so $s'_i \in S''_i$. □

(i) By definition appropriate transfinite sequences $(G'_\alpha)_{\alpha \leq \gamma}$ and $(G''_\alpha)_{\alpha \leq \gamma}$ such that $G = G'_0 = G''_0$, $G' = G'_\gamma$ and $G'' = G''_\gamma$ exist. We proceed by transfinite induction.

Suppose the claim holds for all $\beta < \gamma$.

Case 1. γ is a successor ordinal, say $\gamma = \beta + 1$.

By the induction hypothesis $G'_\beta \subseteq G''_\beta$. By Claim 1 $G'_\gamma \subseteq G_2$, where $G''_\beta \overset{f}{\rightsquigarrow} G_2$. But by the definition of the $\overset{f}{\rightsquigarrow}$ reduction also $G_2 \subseteq G''_\gamma$. So $G'_\gamma \subseteq G''_\gamma$.

Case 2. γ is a limit ordinal.

By the induction hypothesis for all $\beta < \gamma$ we have $G'_\beta \subseteq G''_\beta$. By definition $G'_\gamma = \bigcap_{\beta < \gamma} G'_\beta$ and $G''_\gamma = \bigcap_{\beta < \gamma} G''_\beta$, so $G'_\gamma \subseteq G''_\gamma$.

(ii) Let $(G_\alpha)_{\alpha \leq \beta}$ and $(G'_\alpha)_{\alpha \leq \gamma}$ be the sequences of the reduction of H that respectively ensure $H \overset{f}{\rightsquigarrow}^{\beta} G$ and $H \rightsquigarrow^{\gamma} G$.

Suppose now that on the contrary $\gamma < \beta$. Then $G_\beta \subset G_\gamma$ by the definition of the $\overset{f}{\rightsquigarrow}$ reduction. By (i) we also have $G_\gamma \subseteq G'_\gamma$. Further, $G'_\gamma = G_\beta$ since by assumption both of them equal G, so $G_\beta = G_\gamma$, which is a contradiction. □

It is important to note that the outcome of the considered iterated elimination process can be an empty game.

Example 4.4 Consider a two-players game H in which the set of strategies for each player is the set of natural numbers. The payoff to each player is the number (strategy) he selected. Suppose that beliefs are the strategies of the opponent. Clearly no strategy is a best response to a strategy of the opponent. So $H \rightsquigarrow \emptyset_2$. □

In general, infinite sequences of the $\rightsquigarrow$ reductions are possible. Even more, in some games ω steps of the $\overset{f}{\rightsquigarrow}$ reduction are insufficient to reach a $\mathcal{B}$-closed game.

Example 4.5 Consider the following game H with three players. The set of strategies for each player is the set of natural numbers $\mathcal{N}$. The payoff functions are defined as follows:

$$p_1(k, \ell, m) := \begin{cases} k & \text{if } k = \ell + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$p_2(k, \ell, m) := \begin{cases} k & \text{if } k = \ell \\ 0 & \text{otherwise} \end{cases}$$

$$p_3(k, \ell, m) := 0.$$

Further we assume the pure belief structure. Each restriction of H can be identified with the triple of the strategy sets of the players. Note that

- the best response to $s_{-1} = (\ell, m)$ is $\ell + 1$,
- the best response to $s_{-2} = (k, m)$ is k,
- each $m \in \mathcal{N}$ is a best response to $s_{-3} = (k, \ell)$.

So the following sequence of reductions holds:

$$(\mathcal{N}, \mathcal{N}, \mathcal{N}) \stackrel{f}{\leadsto} (\mathcal{N} \setminus \{0\}, \mathcal{N}, \mathcal{N}) \stackrel{f}{\leadsto} (\mathcal{N} \setminus \{0\}, \mathcal{N} \setminus \{0\}, \mathcal{N}) \stackrel{f}{\leadsto}$$

$$(\mathcal{N} \setminus \{0,1\}, \mathcal{N} \setminus \{0\}, \mathcal{N}) \stackrel{f}{\leadsto} (\mathcal{N} \setminus \{0,1\}, \mathcal{N} \setminus \{0,1\}, \mathcal{N}) \stackrel{f}{\leadsto} \ \ldots$$

So $(\mathcal{N}, \mathcal{N}, \mathcal{N}) \stackrel{f}{\leadsto}^{\omega} (\emptyset, \emptyset, \mathcal{N})$. Also $(\emptyset, \emptyset, \mathcal{N}) \stackrel{f}{\leadsto} (\emptyset, \emptyset, \emptyset)$, so $(\mathcal{N}, \mathcal{N}, \mathcal{N}) \stackrel{f}{\leadsto}^{\omega+1} (\emptyset, \emptyset, \emptyset)$.

Further, it is easy to see that it is the only maximal sequence of the $\leadsto$ reductions. $\square$

Let us mention here that Lipman [1994] constructed a two-player game for which ω steps of the $\stackrel{f}{\leadsto}$ reduction are not sufficient to reach a $\mathcal{B}$-closed game, where each $\mathcal{B}_i$ consists of the mixed strategies of the opponent.

These examples bring us to the question: are we studying the right reduction relation?

5 Variations of the reduction relation

Indeed, a careful reader may have noticed that we use a slightly different notion of reduction than the one considered in Bernheim [1984] and Pearce [1984]. In general, two natural alternatives to the $\leadsto$ relation exist. In this section we introduce these variations and clarify when they coincide.

Given a restriction $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ of a game $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$, a belief structure $(\mathcal{B}, \dot{\cap})$ in H, where $\mathcal{B} := (\mathcal{B}_1, \ldots, \mathcal{B}_n)$ and a restriction $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$ of G, the $\leadsto$ reduction can be alternatively defined by:

- $G \leadsto G'$ when $G \neq G'$ and for all $i \in [1..n]$

$$\forall s_i \in S_i \setminus S'_i \, \forall \mu_i \in \mathcal{B}_i \,\dot{\cap}\, G \, \exists s'_i \in T_i \, p_i(s'_i, \mu_i) > p_i(s_i, \mu_i).$$

Two natural alternatives are:

- $G \rightarrow G'$ when $G \neq G'$ and for all $i \in [1..n]$

$$\forall s_i \in S_i \setminus S'_i \, \forall \mu_i \in \mathcal{B}_i \,\dot{\cap}\, G \, \exists s'_i \in S_i \, p_i(s'_i, \mu_i) > p_i(s_i, \mu_i),$$

- $G \Rightarrow G'$ when $G \neq G'$ and for all $i \in [1..n]$

$$\forall s_i \in S_i \setminus S'_i \, \forall \mu_i \in \mathcal{B}_i \,\dot{\cap}\, G \, \exists s'_i \in S'_i \, p_i(s'_i, \mu_i) > p_i(s_i, \mu_i).$$

So in these two alternatives we refer to better responses in, respectively, G and in G' instead of in H.

Clearly $G \Rightarrow G'$ implies $G \rightarrow G'$ which implies $G \leadsto G'$. However, the reverse implications do not need to hold. The following example additionally shows that neither $\rightarrow$ nor $\Rightarrow$ is order independent. Moreover, countable applications of each of these two relations can reduce the initial game to an empty game.

Example 5.1 Reconsider the two-players game H from Example 4.4. Recall that the set of strategies for each player in H is the set of natural numbers $\mathcal{N}$ and the payoff to each player is the number (strategy) he selected. Also, we assume the pure belief structure.

Given two subsets A_1, A_2 of the set of natural numbers denote by (A_1, A_2) the restriction of H in which A_i is the set of strategies of player i. Clearly for all $k \geq 0$ we have $H \rightsquigarrow (\{k\}, \{k\}) \rightsquigarrow \emptyset_2$ and $H \rightarrow (\{k\}, \{k\})$ and for no $k \geq 0$ and G we have $(\{k\}, \{k\}) \rightarrow G$.

So the relations $\rightsquigarrow$ and $\rightarrow$ differ in the iterations starting at H. Moreover, $\rightarrow$ is not order independent.

Further, for no $k \geq 0$ we have $H \Rightarrow (\{k\}, \{k\})$, so the relations $\rightarrow$ and $\Rightarrow$ differ, as well. Let now for $k \geq 0$

$$(k, \infty) := \{\ell \in \mathcal{N} \mid \ell > k\},$$
$$A_k := \{0\} \cup (k, \infty),$$
$$B_k := \{1\} \cup (k, \infty).$$

Then both

$$H \Rightarrow (A_1, A_1) \Rightarrow (A_2, A_2) \Rightarrow \ \ldots$$

and

$$H \Rightarrow (B_1, B_1) \Rightarrow (B_2, B_2) \Rightarrow \ \ldots,$$

so both $H \Rightarrow^{\omega} (\{0\}, \{0\})$ and $H \Rightarrow^{\omega} (\{1\}, \{1\})$. But for no G and $k \geq 0$ we have $(\{k\}, \{k\}) \Rightarrow G$. This shows that $\Rightarrow$ is not order independent either.

Finally, note that

$$H \Rightarrow ((0, \infty), (0, \infty)) \Rightarrow ((1, \infty), (1, \infty)) \Rightarrow \ \ldots$$

so $H \Rightarrow^{\omega} \emptyset_2$ and hence $H \rightarrow^{\omega} \emptyset_2$, as well. In fact, we also have $H \rightarrow \emptyset_2$. □

Let us define now the counterpart ${}^f\!\!\rightarrow$ of the ${}^f\!\!\rightsquigarrow$ reduction by putting for a restriction $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ of H and a restriction $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$ of G

- $G \ {}^f\!\!\rightarrow G'$ when $G \neq G'$ and for all $i \in [1..n]$

$$S'_i = \{s_i \in S_i \mid \exists \mu_i \in \mathcal{B}_i \,\dot{\cap}\, G \,\forall s'_i \in S_i \; p_i(s'_i, \mu_i) \leq p_i(s_i, \mu_i)\}.$$

So, unlike in the definition of the ${}^f\!\!\rightsquigarrow$ relation, we now refer to the best responses in the game G. Note that $G \ {}^f\!\!\rightarrow G'$ and $G \rightarrow G''$ implies $G' \subseteq G''$. In Bernheim [1984] and Pearce [1984] the ${}^f\!\!\rightarrow$ reduction was studied, in each paper for a specific belief structure.

Observe that the corresponding 'fast' reduction ${}^f\!\!\Rightarrow$ does not exist. Indeed, in the above example we have $H \Rightarrow ((k, \infty), (k, \infty))$ for all $k \geq 0$. But $\bigcap_{k=0}^{\infty} (k, \infty) = \emptyset$ and $H \Rightarrow \emptyset_2$ does not hold. So no G' exists such that $H \Rightarrow G'$ and for all G'', $G \Rightarrow G''$ implies $G' \subseteq G''$.

In the game used above we have both $H \ {}^f\!\!\rightsquigarrow \emptyset_2$ and $H \ {}^f\!\!\rightarrow \emptyset_2$, so both fast reductions coincide when started at H. The next example shows that this is not the case in general. Moreover, it demonstrates that for the $\rightarrow$ relation a stronger reduction can be achieved if non-fast reductions are allowed. So the counterpart of Theorem 4.3 does not hold for the $\rightarrow$ relation.

Example 5.2 Consider a version of Bertrand competition between two firms in which the marginal costs are 0 and in which the range of possible prices is the left-open real interval $(0, 100]$. So in this game H there are two players, each with the set $(0, 100]$ of strategies. We assume that the demand equals $100 - p$, where p is the lower price and that the profits are split in case of a tie. So the payoff functions are defined by:

$$p_1(s_1, s_2) := \begin{cases} s_1(100 - s_1) & \text{if } s_1 < s_2 \\ \dfrac{s_1(100 - s_1)}{2} & \text{if } s_1 = s_2 \\ 0 & \text{if } s_1 > s_2 \end{cases}$$

$$p_2(s_1, s_2) := \begin{cases} s_2(100 - s_2) & \text{if } s_2 < s_1 \\ \dfrac{s_2(100 - s_2)}{2} & \text{if } s_1 = s_2 \\ 0 & \text{if } s_2 > s_1 \end{cases}$$

Also, we assume the pure belief structure. Below we identify the restrictions of H with the pairs of the strategy sets of the players.

Since $s_1 = 50$ maximizes the value of $s_1(100 - s_1)$ in the interval $(0, 100]$, the strategy 50 is the unique best response to any strategy $s_2 > 50$ of the second player. Further, no strategy is a best response to a strategy $s_2 \leq 50$. By symmetry the same holds for the strategies of the second player. So $H \overset{f}{\rightsquigarrow} (\{50\}, \{50\})$. Next, $s_1 = 49$ is a better response in H to $s_2 = 50$ than $s_1 = 50$ and symmetrically for the second player. So $(\{50\}, \{50\}) \overset{f}{\rightsquigarrow} \emptyset_2$.

We also have $H \overset{f}{\rightarrow} (\{50\}, \{50\})$. But $s_1 = 50$ is a best response in $(\{50\}, \{50\})$ to $s_2 = 50$ and symmetrically for the second player. So for no restriction G of H we have $(\{50\}, \{50\}) \rightarrow G$ or $(\{50\}, \{50\}) \overset{f}{\rightarrow} G$. However, we also have $H \rightarrow ((0, 50], (0, 50]) \rightarrow \emptyset_2$. So H can be reduced to the empty game using the $\rightarrow$ reduction but only if non-fast reductions are allowed.

Finally, note that also $H \Rightarrow ((0, 50], (0, 50])$ holds. Let $(r_i)_{i<\omega}$ be a strictly descending sequence of real numbers starting with $r_0 = 50$ and converging to 0. It is easy to see that for $i \geq 0$ we then have $((0, r_i], (0, r_i]) \Rightarrow ((0, r_{i+1}], (0, r_{i+1}])$, so $H \Rightarrow^\omega \emptyset_2$. □

To analyze the situation when the three considered reduction relations coincide we introduce the following property:

B For all restrictions G of H and all beliefs $\mu_i \in \mathcal{B}_i \dot{\cap} G$ a best response to μ_i in G exists.

For the finite games property **B** obviously holds. However, it can fail for infinite games. For instance, it does not hold in the game considered in Examples 4.4 and 5.1 since in this game no strategy is a best response to a strategy of the opponent.

In the presence of property **B** the reductions $\rightarrow$ and $\Rightarrow$ are equivalent.

Lemma 5.3 (Equivalence) *Consider a game H and a belief structure $(\mathcal{B}, \dot{\cap})$ in H. Assume property **B**. The relations $\rightarrow$ and $\Rightarrow$ coincide on the set of restrictions of H.*

Proof. Clearly if $G \Rightarrow G'$, then $G \rightarrow G'$. To prove the converse let $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$, $G' := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$ and $\mathcal{B} := (\mathcal{B}_1, \ldots, \mathcal{B}_n)$.

Suppose $G \rightarrow G'$. Take an arbitrary $s_i \in S_i \setminus S'_i$ and an arbitrary $\mu_i \in \mathcal{B}_i \dot{\cap} G$. By property **B** some $s'_i \in S_i$ is a best response to μ_i in G. By definition this s'_i is not eliminated in the step $G \rightarrow G'$, i.e., $s'_i \in S'_i$. So s_i is not a best response to μ_i in G'. This proves $G \Rightarrow G'$. □

However, the situation changes when we consider the $\rightsquigarrow$ relation. We noted already that $G \rightarrow G'$ implies $G \rightsquigarrow G'$. But the converse does not need to hold, even if property **B** holds.

Example 5.4 Suppose that H equals

	L	R
T	$2,0$	$2,0$
M	$0,0$	$1,0$
B	$1,0$	$0,0$

G is

	L	R
M	$0,0$	$1,0$
B	$1,0$	$0,0$

and G' is

	L	R
M	$1,0$	$1,0$

Further, assume the pure belief structure. Property **B** holds since the game H is finite.

Since the strategy B is never a best response to a strategy of the opponent in the game H, we have $G \rightsquigarrow G'$ but $G \rightarrow G'$ does not hold since B is a best response to L in the game G. $\Box$

On the other hand, in the presence of properties **A** and **B**, iterated applications of the $\rightsquigarrow$ reduction started in H do yield the same outcome as the iterated applications of $\rightarrow$ or of $\Rightarrow$. Indeed, the following holds.

Lemma 5.5 (Equivalence) *Consider a game H and a belief structure $(\mathcal{B}, \dot{\cap})$ in H. Assume properties **A** and **B**. For all restrictions G of H, $H \rightsquigarrow^{\gamma} G$ iff $H \rightarrow^{\gamma} G$.*

Proof. Since $G' \rightarrow G''$ implies $G' \rightsquigarrow G''$, for all γ $H \rightarrow^{\gamma} G$ implies $H \rightsquigarrow^{\gamma} G$.

To prove the converse we proceed by transfinite induction. Assume that $H \rightsquigarrow^{\gamma} G$. By definition an appropriate transfinite sequence of restrictions $(G_\alpha)_{\alpha \leq \gamma}$ of H with $H = G_0$ and $G_\gamma = G$ exists ensuring that $H \rightsquigarrow^{\gamma} G$.

Suppose the claim of the lemma holds for all $\beta < \gamma$.

Case 1. γ is a successor ordinal, say $\gamma = \beta + 1$.

Then $H \rightsquigarrow^{\beta} G_\beta$ and $G_\beta \rightsquigarrow G$. Suppose that $\mathcal{B} := (\mathcal{B}_1, \ldots, \mathcal{B}_n)$, $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$, $G_\beta := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ and $G := (S'_1, \ldots, S'_n, p_1, \ldots, p_n)$.

Consider an arbitrary $s_i \in S_i \setminus S'_i$ and an arbitrary $\mu_i \in \mathcal{B}_i \dot{\cap} G_\beta$ such that s_i is not a best response in H to μ_i. By property **A**

$$\mu_i \in \mathcal{B}_i \dot{\cap} G_\alpha \text{ for all } \alpha \leq \beta. \tag{1}$$

By property **B** a best response s'_i to μ_i in H exists. Then $p_i(s'_i, \mu_i) > p_i(s_i, \mu_i)$ and $p_i(s'_i, \mu_i) \geq p_i(s''_i, \mu_i)$ for all $s''_i \in T_i$. By the latter inequality and (1) s'_i is not removed in any $\rightsquigarrow$ step leading from H to G_β. So $s'_i \in S_i$ and by the former (strict) inequality s_i is not a best response to μ_i in G_β. This proves $G_\beta \rightarrow G$. But by the induction hypothesis $H \rightarrow^{\beta} G_\beta$, so $H \rightarrow^{\gamma} G$.

Case 2. γ is a limit ordinal.

By the induction hypothesis for all $\beta < \gamma$ we have $H \rightsquigarrow^{\beta} G_\beta$ iff $H \rightarrow^{\beta} G_\beta$, so by definition $H \rightsquigarrow^{\gamma} G$ iff $H \rightarrow^{\gamma} G$. $\Box$

This allows us to establish an order independence result for the $\rightarrow$ and $\Rightarrow$ relations.

Theorem 5.6 (Order Independence) *Consider a game H and a belief structure $(\mathcal{B}, \dot{\cap})$ in H. Assume properties **A** and **B**.*

(i) All maximal sequences of the $\rightarrow$, $\Rightarrow$ and $\rightsquigarrow$ reductions yield the same outcome G.

(ii) This restriction G satisfies the following property:

each strategy s_i of player i in G is a best response in G (note this reference to G and not H) to a belief in $\mathcal{B}_i \dot{\cap} G$.

Proof.
(i) By the Order Independence Theorem 4.2 and

(ii) By (i) for no G' we have $G \rightarrow G'$, which proves the claim. □

6 Beliefs as joint pure strategies of the opponents

So far we established results for arbitrary belief structures that satisfy properties **A** and **B**. In this section we analyze what additional properties hold for the case of pure belief structures. So given a game $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$ we assume $\mathcal{B}_i := T_{-i}$ for $i \in [1..n]$ and for a restriction $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ of H we assume $T_{-i} \dot{\cap} G := S_{-i}$.

Clearly, property **A** then holds. By the Order Independence Theorem 4.2, the outcome of each maximal sequence of the $\rightsquigarrow$ reductions is unique. We noted already that this outcome can be an empty game. On the other hand, if the initial game has a Nash equilibrium, then this unique outcome cannot be a degenerate restriction. Indeed, the following result holds.

Theorem 6.1 *Consider a game $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$. Suppose that $H \rightsquigarrow^{\gamma} G$ for some γ.*

(i) If s is a Nash equilibrium of H, then it is a Nash equilibrium of G. Consequently, if G is empty, then H has no Nash equilibrium.

(ii) Suppose that for each $s_{-i} \in T_{-i}$ a best response to s_{-i} in H exists. If s is a Nash equilibrium of G, then it is a Nash equilibrium of H.

Proof.
(i) Let G' be the unique outcome of a maximal sequence of the $\rightsquigarrow$ reductions that starts with $H \rightsquigarrow^{\gamma} G$. By definition $(s_1, \ldots, s_n)$ is a Nash equilibrium of H iff each s_i is a best response to s_{-i} iff (by the choice of $\mathcal{B}$) $(\{s_1\}, \ldots, \{s_n\})$ is $\mathcal{B}$-closed. Hence, by the Order Independence Theorem 4.2, each Nash equilibrium s of H is present in G' and hence in G. But G is a restriction of H, so s is also a Nash equilibrium of G.

(ii) Suppose s is not a Nash equilibrium of H. Then some s_i is not a best response to s_{-i} in H. By assumption a best response s'_i to s_{-i} in H exists. Then

$$p_i(s'_i, s_{-i}) > p_i(s).$$

The strategy s'_i is not eliminated in any $\rightsquigarrow$ step leading from H to G, since s_{-i} is a joint strategy of the opponents of player i in all games in the considered maximal sequence. So s'_i is a strategy of player i in G, which contradicts the fact that s is a Nash equilibrium of G. □

The above result applies to all three reduction relations since $H \rightarrow^{\gamma} G$ implies $H \rightsquigarrow^{\gamma} G$ and $H \Rightarrow^{\gamma} G$ implies $H \rightsquigarrow^{\gamma} G$.

The assumption used in (ii) is implied by property **B**. A natural situation when property **B** holds is the following. We call a game $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$ ***compact*** if the strategy sets are non-empty compact subsets of a complete metric space and ***own-uppersemicontinuous*** if each payoff function p_i is uppersemicontinuous in the ith argument.[2]

As explained in Dufwenberg and Stegeman [2002] (see the proof of Lemma on page 2012) for such games property **B** holds by virtue of a standard result from topology. Consequently, by the Order Independence Theorem 5.6, the order independence for the $\rightarrow$, $\Rightarrow$ and $\rightsquigarrow$ reduction relations holds. Let us also mention that for this class of games Dufwenberg and Stegeman [2002] established order independence of the iterated elimination of strictly dominated strategies.

If we impose a stronger condition on the payoff functions, namely that each of them is continuous, then we are within the framework considered in Bernheim [1984]. As shown in this paper if at each stage the ${}^{f}\!\!\rightarrow$ reduction is applied, the final (unique) outcome is a non-degenerate restriction and is reached after at most ω steps. This allows us to draw the following corollary to the Order Independence Theorems 4.2 and 5.6.

Corollary 6.2 *Consider a compact game H with continuous payoff functions. All maximal sequences of the $\rightsquigarrow$ (or $\rightarrow$ or $\Rightarrow$) reductions starting in H yield the same outcome which is a non-degenerate restriction of H.* □

If at each stage only some strategies that are NBR are removed, transfinite reduction sequences of length $> \omega$ are possible. In Section 4 we already noted that in some games such transfinite sequences are unavoidable.

Let us mention here that under the same assumptions about the game H Ambroszkiewicz [1994] showed the analogue of the Equivalence Lemma 5.5 for the 'fast' counterparts ${}^{f}\!\!\rightsquigarrow$ and ${}^{f}\!\!\rightarrow$ of the reduction relations $\rightsquigarrow$ and $\rightarrow$, for the limited case of two-person games and beliefs equal to the strategies of the opponent. [3] By the abovementioned result of Bernheim [1984] the corresponding iterations of these two reduction relations reach the final outcome after at most ω steps.

Recall now that a simple strengthening of the assumptions of Bernheim [1984] leads to a framework in which existence of a (pure) Nash equilibrium is ensured. Namely, assume that strategy sets are non-empty compact convex subsets of a complete metric space and each payoff function p_i is continuous and quasi-concave in the ith argument.[4] By a theorem of Debreu [1952], Fan [1952] and Glicksberg [1952] under these assumptions a Nash equilibrium exists.

Natural examples of games satisfying these assumptions are ***mixed extensions*** of finite games, i.e., games in which the players' strategies are their mixed strategies in a finite game H and the payoff functions are the canonic extensions of the payoffs in H to the joint mixed strategies.

Let us modify now the definition of the narrowing operation $\dot{\cap}$ by putting for a mixed extension $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$ and its restriction $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$

$$T_{-i} \,\dot{\cap}\, G := \Pi_{j \neq i} \overline{S_j}, \tag{2}$$

[2] Recall that p_i is ***uppersemicontinuous in the ith argument*** if the set $\{s'_i \in T_i \mid p_i(s'_i, s_{-i}) \geq r\}$ is closed for all $r \in \mathcal{R}$ and all $s_{-i} \in T_{-i}$.

[3] To be precise, in his definition the 'fast' reductions are defined by considering the reduction for each player in succession and not in parallel.

[4] Recall that p_i is ***quasi-concave in the ith argument*** if the set $\{s'_i \in T_i \mid p_i(s'_i, s_{-i}) \geq p_i(s)\}$ is convex for all $s \in T$.

where for a set M_j of mixed strategies of player j $\overline{M_j}$ denotes its convex hull. Then, as before, properties **A** and **B** hold.

This situation corresponds to the setup of Pearce [1984] in which at each stage *all* mixed strategies that are NBR are deleted and $\dot{\cap}$ is defined by (2). Pearce [1984] proved that this iterative process based on the $\stackrel{f}{\rightarrow}$ reduction terminates after finitely many steps and yields a non-degenerate restriction. So we get another corollary to the Order Independence Theorem 4.2 and the Equivalence Lemmata 5.3 and 5.5.

Corollary 6.3 *Let $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$ be a mixed extension of a finite game. Suppose that $\dot{\cap}$ is defined by (2). Then all maximal sequence of the $\rightsquigarrow$ (or $\rightarrow$ or $\Rightarrow$) reductions yield the same outcome which is a non-degenerate restriction of H.* □

The same outcome is obtained when at each stage only some mixed strategies that are NBR are deleted. In this case the iteration process can be infinite, possibly continuing beyond ω.

7 Finite games

Finally, we consider the case of ***finite*** games, i.e., ones in which all strategy sets are finite. Given a finite non-empty set A we denote by ΔA the set of probability distributions over A.

Consider a finite game $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$. In what follows by a ***belief of player*** i in the game H we mean a probability distribution over the set of joint strategies of his opponents. So ΔT_{-i} is the set of beliefs. The payoff functions p_i are modified to the expected payoff functions in the standard way by putting for $\mu_i \in \Delta T_{-i}$:

$$p_i(s_i, \mu_i) := \sum_{s_{-i} \in T_{-i}} \mu_i(s_{-i}) \cdot p_i(s_i, s_{-i}).$$

We noted already that for the finite games property **B** obviously holds. For further considerations we need the following property:

C For all non-degenerate restrictions G of H, $\mathcal{B}_i \dot{\cap} G \neq \emptyset$.

Note 7.1 *Consider a finite game $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$ and a belief structure $(\mathcal{B}, \dot{\cap})$ in H. Assume properties **A**-**C**. Then a non-degenerate $\mathcal{B}$-closed restriction of H exists.*

Proof. Keep applying the $\rightsquigarrow$ reduction starting with the original game H. Since now only finite sequences of $\rightsquigarrow$ reductions exist, this iteration process stops after finitely many steps. By the Equivalence Lemma 5.5 its outcome coincides with the repeated application of the $\rightarrow$ reduction.

But by definition, in the presence of properties **A**-**C**, if G is a non-degenerate restriction of H and $G \rightarrow G'$, then G' is non-degenerate, as well. So in this iteration process only non-degenerate restrictions are produced. □

Three successively larger sets of beliefs are of interest:

- $\mathcal{B}_i = T_{-i}$ for $i \in [1..n]$.

  Then beliefs are joint pure strategies of the opponents.

- $\mathcal{B}_i = \Pi_{j \neq i} \Delta T_j$ for $i \in [1..n]$.

  Then beliefs are joint mixed strategies of the opponents.

- $\mathcal{B}_i = \Delta T_{-i}$ for $i \in [1..n]$.

  Then beliefs are probability distributions over the set of joint pure strategies of the opponents.

These sets of beliefs are increasingly larger in the sense that we can identify T_{-i} with the subset of $\Pi_{j \neq i} \Delta T_j$ consisting of the joint pure strategies and in turn $\Pi_{j \neq i} \Delta T_j$ with the subset of ΔT_{-i} consisting of the so-called *uncorrelated* beliefs.

For $\mathcal{B}_i \subseteq \Delta T_{-i}$ and $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$ we define then

$$\mathcal{B}_i \mathbin{\dot\cap} G := \{\mu_i \in \mathcal{B}_i \mid \mu_i(s_{-i}) = 0 \text{ for } s_{-i} \in T_{-i} \setminus S_{-i}\}. \tag{3}$$

In particular, in view of the above identifications,

$$T_{-i} \mathbin{\dot\cap} G = S_{-i},$$

$$\Pi_{j \neq i} \Delta T_j \mathbin{\dot\cap} G = \Pi_{j \neq i} \Delta S_j,$$

and

$$\Delta T_{-i} \mathbin{\dot\cap} G = \Delta S_{-i}.$$

Consider now properties **A** and **C**. Property **A** obviously holds. In turn, property **C** holds if $T_{-i} \subseteq \mathcal{B}_i$ for all $i \in [1..n]$.

Summarizing, in view of Note 7.1 we get the following corollary to the Order Independence Theorem 4.2 and the Equivalence Lemmata 5.3 and 5.5.

Corollary 7.2 *Consider a finite game $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$ and suppose that $T_{-i} \subseteq \mathcal{B}_i$ for all $i \in [1..n]$ and that $\dot\cap$ is defined by (3). Then all maximal sequences of the $\rightsquigarrow$ (or $\rightarrow$ or $\Rightarrow$) reductions yield the same outcome which is a non-degenerate restriction of H.* □

In particular, each set $\mathcal{B}_i$ can be instantiated to any of the three sets of beliefs listed above. However, the assumption that $T_{-i} \subseteq \mathcal{B}_i$ for all $i \in [1..n]$ excludes systems of beliefs $\mathcal{B} := (\mathcal{B}_1, \ldots, \mathcal{B}_n)$ that consist of the joint totally mixed strategy of the opponents. Recall that a mixed strategy is called ***totally mixed*** if it assigns a positive probability to each pure strategy. Indeed, any element $s_{-i} \in T_{-i}$ is a sequence of pure strategies of the opponents of player i and each such pure strategy s_j is identified with a mixed strategy that puts all weight on s_j (and hence weight zero on other pure strategies). So no element of s_{-i} from T_{-i} can be identified with a totally mixed strategy.

Observe also that if each $\mathcal{B}_i$ is the set of joint totally mixed strategy of the opponents, then for each proper restriction G of H the sets $\mathcal{B}_i \mathbin{\dot\cap} G$ are all empty. So $\mathcal{B}_i \mathbin{\dot\cap} G$ does not model the set of joint totally mixed strategies of the opponents of player i in the game G.

The systems of beliefs involving totally mixed strategies were studied in several papers, starting with Pearce [1984], where a best response to a belief formed by a joint totally mixed strategy of the opponents is called a ***cautious response***. A number of modifications of the notion of rationalizability rely on a specific use of totally mixed strategies, see, e.g., Herings and Vannetelbosch [2000] where the notion of weak perfect rationalizability is studied.

Corollary 7.2 does not apply to the iterated elimination procedures based on such systems of beliefs. This is not surprising, since as shown in Pearce [1984] a strategy is weakly dominated iff it is never a cautious response, and weak dominance is order dependent. In turn, the elimination procedure discussed in Herings and Vannetelbosch [2000] is shown to be equivalent to the Dekel and Fudenberg [1990] elimination procedure which consists of one round of elimination of all weakly dominated strategies followed by the iterated elimination all strictly dominated strategies.

8 Concluding remarks

We studied in this paper the problem of order independence for rationalizability in strategic games. To this end we relaxed the requirement that at each stage all strategies that are never best responses are eliminated. This brought us to a study of three natural reduction relations.

The iterated elimination of NBR is supposed to model reasoning of a rational player, so we should reflect on the consequences of the obtained results. First, we noted that in some games the transfinite iterations can be unavoidable. This difficulty was already discussed in Lipman [1994] who concluded that finite order mutual knowledge may be insufficient as a characterization of common knowledge.

Next, we noted that in the natural situation when beliefs are the joint pure strategies of the opponents empty games can be generated using each of the reduction relations $\rightsquigarrow$, $\rightarrow$ and $\Rightarrow$. We could interpret such a situation as a statement that in the initial game no player has a meaningful strategy to play. Note that Theorem 6.1 allows us to conclude that the initial game has then no Nash equilibrium.

Another issue is which of the three reduction relations is the 'right' one. The first one considered, $\rightsquigarrow$, is the *strongest* in the sense that its iterated applications achieve the strongest reduction. It is order independent under a very weak assumption **A** that captures the idea of a 'well-behaving' belief structure.

However, its definition refers to the strategies of the initial game H which at the moment of reference may already have been discarded. This point can be illustrated using the Bertrand competition game of Example 5.2. We concluded there that $(\{50\}, \{50\}) \rightsquigarrow \emptyset_2$ because $s_1 = 49$ is a better response of the first player in H to $s_2 = 50$ than $s_1 = 50$ and symmetrically for the second player. However, the strategy $s_1 = 49$ is already discarded at the moment the game $(\{50\}, \{50\})$ is considered, so —one might argue— it should not be used to discard another strategy. If one accepts this viewpoint, then one endorses $\rightarrow$ as the right reduction. This reduction relation is not order independent under assumption **A** but is order independent once we add assumption **B** stating that for each belief μ_i in a restriction G of the original game a best response to μ_i in G exists.

Finally, we can view the $\Rightarrow$ reduction as a 'conservative' variant of $\rightarrow$ in which one insists that the 'witnesses' used to discard the strategies should not themselves be discarded (in the same round). In the case of the iterated elimination of strictly dominated strategies the corresponding reduction relation was studied in Gilboa, Kalai and Zemel [1990] and Dufwenberg and Stegeman [2002].

Note that the difficulty of chosing the right reduction relation does not arise in Bernheim [1984] and Pearce [1984] since for the class of the games there studied properties **A** and **B** hold and consequently the Equivalence Lemmata 5.3 and 5.5 can be applied.

In the previous two sections we established order independence for all three reduction relations $\rightsquigarrow$, $\rightarrow$ and $\Rightarrow$ for the same classes of games for which order independence of the iterated elimination of strictly dominated strategies (SDS) holds. It was already indicated in Pearce [1984] that the iterated elimination of NBR yields a stronger reduction than the iterated elimination of SDS. The Bertrand competition game of Example 5.2 provides an illuminating example of this phenomenon. In this game all three reduction relations $\rightsquigarrow$, $\rightarrow$ and $\Rightarrow$ allow us to reduce the initial game to an empty one. However, in this game no strategy strictly dominates another one. Indeed, for any s_1 and s_1' such that $0 < s_1 < s_1' \leq 100$ we have $p_1(s_1, s_2) = p_1(s_1', s_2) = 0$ for all s_2 such that $0 < s_2 < s_1$ and analogously for the second player. So no strategy can be eliminated on the account of strict dominance.

This advantage of each variant of the iterated elimination of NBR over the iterated elimination of SDS disappears if we provide each player with the strategy 0. Then each strategy is a best response to the strategy 0 of the opponent, since all of them yield the same payoff, 0. So no strategy can be eliminated and all four elimination methods yield no reduction, while the resulting game has a unique Nash equilibrium, namely $(0, 0)$.

Let us conclude with an example when two variants of the iterated elimination of NBR allow one to identify the unique Nash equilibrium, while the iterated elimination of SDS yields no reduction.

Example 8.1 Consider Hotelling location game in which two sellers choose a location in the open real interval $(0, 100)$. So in this game H there are two players, each with the set $(0, 100)$ of strategies. The payoff functions p_i $(i = 1, 2)$ are defined by:

$$p_i(s_i, s_{3-i}) := \begin{cases} s_i + \dfrac{s_{3-i} - s_i}{2} & \text{if } s_i < s_{3-i} \\ 100 - s_i + \dfrac{s_i - s_{3-i}}{2} & \text{if } s_i > s_{3-i} \\ 50 & \text{if } s_i = s_{3-i} \end{cases}$$

First note that no strategy strictly dominates another one. Indeed, for any s_1 and s'_1 such that $0 < s_1 < s'_1 < 100$ we have $p_1(s_1, s_2) < p_1(s'_1, s_2)$ for all s_2 such that $s'_1 < s_2 < 100$ and $p_1(s_1, s_2) > p_1(s'_1, s_2) = 0$ for all s_2 such that $0 < s_2 < s_1$. A symmetric reasoning holds for the second player.

Next, we consider the reduction relations $\rightsquigarrow$, $\rightarrow$ and $\Rightarrow$ defined as in Section 6. Note that no strategy $s_1 \in (0, 100) \setminus \{50\}$ is a best response in H to a strategy $s_2 \in (0, 100)$. Indeed, if $s_1 \neq s_2$, then we have $p_1(s_1, s_2) < p_1(s'_1, s_2)$ for all s'_1 such that $s'_1 \in (min(s_1, s_2), max(s_1, s_2)$. And if $s_1 = s_2$, then by assumption $s_1 \neq 50$ and we have then $p_1(s_1, s_2) = 50 < p_1(50, s_2)$. A symmetric reasoning holds for the second player.

So both $H \rightsquigarrow (\{50\}, \{50\})$ and $H \rightarrow (\{50\}, \{50\})$ and $(\{50\}, \{50\})$ is Nash equilibrium of H. Also note that $\Rightarrow$ yields no reduction here. $\Box$

Acknowledgements

We thank one of the referees for useful comments and for drawing our attention to the paper Ambroszkiewicz [1994].

References

S. Ambroszkiewicz
[1994] Knowledge and best responses in games, *Annals of Operations Research*, 51, pp. 63–71.

B. D. Bernheim
[1984] Rationalizable strategic behavior, *Econometrica*, 52, pp. 1007–1028.

G. Debreu
[1952] A social equilibrium existence theorem, *Proceedings of the National Academy of Sciences*, 38, pp. 886–893.

E. Dekel and D. Fudenberg
[1990] Rational behavior and payoff uncertainty, *Journal Of Economic Theory*, pp. 1391–1402.

M. Dufwenberg and M. Stegeman

[2002] Existence and uniqueness of maximal reductions under iterated strict dominance, *Econometrica*, 70, pp. 2007–2023.

K. Fan

[1952] Fixed point and minimax theorems in locally convex topological linear spaces, *Proceedings of the National Academy of Sciences*, 38, pp. 121–126.

I. Gilboa, E. Kalai, and E. Zemel

[1990] On the order of eliminating dominated strategies, *Operation Research Letters*, 9, pp. 85–89.

I. L. Glicksberg

[1952] A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points, *Proceedings of the American Mathematical Society*, 3, pp. 170–174.

P. J.-J. Herings and V. J. Vannetelbosch

[2000] The equivalence of the Dekel-Fudenberg iterative procedure and weakly perfect rationalizability, *Economic Theory*, pp. 677–687.

B. L. Lipman

[1994] A note on the implications of common knowledge of rationality, *Games and Economic Behavior*, 6, pp. 114–129.

M. J. Osborne and A. Rubinstein

[1994] *A Course in Game Theory*, The MIT Press, Cambridge, Massachusetts.

D. G. Pearce

[1984] Rationalizable strategic behavior and the problem of perfection, *Econometrica*, 52, pp. 1029–1050.

M. Stegeman

[1990] *Deleting strictly eliminating dominated strategies.* Working Paper 1990/6, Department of Economics, University of North Carolina.