On Epistemic Logic with Justification

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Abstract

The *true belief* components of Plato's tripartite definition of knowledge as justified true belief are represented in formal epistemology by modal logic and its possible worlds semantics. At the same time, the *justification* component of Plato's definition did not have a formal representation. This paper introduces the notion of justification into formal epistemology. Epistemic logic with justification, along with the usual knowledge operator $\Box F$ (F *is known*), contains assertions $t{:}F$ (t *is a justification for* F). We suggest an epistemic semantics which augments Kripke models with a natural Fitting-style treatment of justification assertions $t{:}F$. Completeness and some new specific properties of basic systems of epistemic logic with justification are established.

1 Introduction

Plato's much celebrated tripartite definition of knowledge as *justified true belief* (JTB) is generally regarded as a set of necessary conditions for the possession of knowledge. Due to Hintikka, the "true belief" components have been fairly formalized by means of modal logic and its possible worlds semantics. Despite the fact that the justification condition has received the greatest attention in epistemology (cf., for example, [12, 22, 26, 27, 32, 33, 34, 40]), it lacked a formal representation. The issue of finding a formal epistemic logic with justification has been discussed in [44]. Such a logic should contain assertions of the form $\Box F$ (F *is known*), along with those of the form $t{:}F$ (t *is a justification for* F).

We introduce justification into formal epistemology by combining Hintikka-style epistemic modal logic with justification calculi arising from the logic of proofs ([2, 3, 4]). In particular, we consider natural combinations of epistemic modal logic $\mathsf{S4}$ with the logic of proofs LP. However, this approach is flexible with respect to both the knowledge/belief component for $\Box F$ and the justification component for $t{:}F$, which can be chosen independently.

Epistemic systems with justification based on the logic of proofs LP use the following plausible assumptions: 1) each axiom has justification; 2) justification is checkable; 3) justification assertion of a statement implies knowledge of this statement; 4) any justification is compatible with any other justification. There are other known justification systems ([8, 10, 11]), each capturing its own

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set of justification properties; this opens the way to a variety of systems for epistemic logic with justification.

Formalization of justification significantly expands the expressive power of epistemic logic and provides a new tool for formal studies in epistemology and applications. Here are some epistemological notions which seem to be affected by this new development.

1. The foundational *Gettier problem* of augmenting the tripartite JTB definition of knowledge (cf., for example, [12, 22, 26, 27, 32, 33, 34, 40]) becomes a formal epistemology issue. Within the basic epistemic systems introduced in this paper, the Gettier counterexamples fail because of assumption 3), which yields that any "justified" sentence should be true. This may be regarded as a kind of a "no false lemmas" formal solution to the Gettier problem. However, we are not taking sides in this issue: the proposed framework provides formal tools for exploring other solutions of the Gettier problem as well.

2. The traditional Hintikka-style modal logic approach to knowledge has the well-known defect of *logical omniscience*, caused by an unrealistic stipulation that an agent knows all logical consequences of his/her assumptions ([16, 36, 41, 42]). The usual epistemic modality $\Box F$ should be regarded as "potential knowledge" or "knowability" (cf. [20]) rather than actual knowledge. Epistemic systems with justification address the issue of logical omniscience in a natural way. A justified knowledge $t{:}F$ cannot be asserted without presenting an explicit justification t for F, hence justified knowledge is not logically omniscient[1].

3. Epistemic logic with justifications was used in [6] to offer a new approach to *common knowledge*. A new modal operator $J\varphi$ for *justified knowledge* introduced in [6] is defined as a forgetful projection of justification assertions $t{:}\varphi$ in a multi-agent epistemic logic with common justification. It turned out that justified knowledge is a special constructive version of common knowledge and can be used as such in solving specific problems. Justified knowledge is considerably more flexible and in many respects easier than the traditional common knowledge.

4. There is one more issue which is naturally handled in the epistemic logic with justification: an intensional and extensional representation of knowledge. Knowledge statements "F is known" ($\Box F$) remain *extensional*, as in Hintikka's logic of knowledge, whereas new justification statements $t{:}F$ are already *intensional*. Indeed, the facts that $t{:}F$ holds and G is (even provably) equivalent to F do not yield $t{:}G$ as well. If there is a justification s for $F \to G$, then a justification for G is a certain function of s and t, which is, generally speaking, different from t. Formal axioms and rules of epistemic logic with justification capture this distinction.

5. Justification enables us to formulate *new epistemic principles*. For example, in the context of mathematical provability, the modal principle of negative introspection $\neg\Box F \to \Box\neg\Box F$ is not valid. A purely explicit version of negative introspection $\neg(x{:}F) \to t(x){:}\neg(x{:}F)$ does not hold in the logic of proofs LP either. However, negative introspection in a mixed explicit-implicit form $\neg(t{:}F) \to \Box\neg(t{:}F)$ is valid in the provability semantics ([2]), which provides a good reason for considering this principle in the general epistemic context.

2 The origins of epistemic logic with justification

The logic of proofs LP was inspired by the classical works of the 1930s by Kolmogorov [29] and Gödel ([23, 24]) and found in [3, 4] (see also surveys [5, 7, 15]). LP describes all valid principles of

[1] This property was formally established for the logic of proofs LP in a recent work by S. Artemov and R. Kuznets: for any valid knowledge assertion $t{:}F$ there exists a proof for F which length is linear in the length of $t{:}F$.

proof operators $t{:}F$

$$t \text{ is a proof of } F \text{ in Peano arithmetic} \tag{1}$$

with an appropriate set of operations on proofs sufficient to realize the whole of S4 explicitly. A similar explicit counterpart of S5 was found in [8]. Explicit versions of K, T, D, K4, and D4 were found in [10, 11][2].

A semantical approach to the logic of proofs has been developed in the papers by Mkrtychev [35] and Fitting [17, 19, 20], where the semantics of justification assertions $t{:}F$ as "F holds and t is a justification for F" was coined.

Joint logics of proofs and provability, studied in [2, 9, 37, 38, 39, 43, 45], are of special interest for the purposes of this paper, since they serve as prototypes of epistemic logic with justification.

In this paper, we study two systems of epistemic logics with justification. The basic one, S4LP, consists of S4 combined with LP as a calculus of justification and the principle $t{:}F \to \Box F$ connecting justification with knowledge. The other system, S4LPN, is S4LP augmented by the principle of *explicit negative introspection* $\neg(t:F) \to \Box\neg(t:F)$ which first came up in the logics of proofs and provability [2]. We establish soundness and completeness theorems for S4LP and S4LPN with respect to AF-semantics. Furthermore, both S4LP and S4LPN are shown to enjoy the arithmetical provability semantics when $\Box F$ is interpreted as the so-called *strong provability operator* (cf. [14]):

$$F \text{ is true and provable in Peano arithmetic} \ .$$

Systems S4LP and S4LPN were introduced in our technical report [9], where Fitting models of the logic of proofs were adopted for epistemic logic with justification. Soundness of S4LP with respect to F-models, Theorems 3 and 4 of the current paper were established in [9]. A question of completeness of S4LP with respect to F-models was formulated there as an open problem. Later in [18], Fitting answered this question and established the desired completeness result. A general notion of AF-model covering all known systems of epistemic logic with justification, S4LP and S4LPN including, was introduced by Artemov in [6].

Both F-models and AF-models make sense for multi-agent epistemic logics with justification. F-models describe relationships between justification assertions $t{:}\varphi$ and their forgetful projection, the *justified knowledge* modality $J\varphi$ introduced in [6]. The intended reading of $J\varphi$ is "justification of φ is available." In a formal model, J is the modality whose accessibility relation coincides with the evidence accessibility relation R^e. AF-models describe relationships between justification assertions $t{:}\varphi$ and knowledge of an arbitrary agent who respects evidence, but whose knowledge may go beyond the strictly justified one (cf. [6]).

3 The logic of proofs as a general calculus of justification

The logic of proofs naturally extends classical propositional logic by adding symbolically represented proofs into the language of the system. Internal proof terms in LP are called proof polynomials. A new formula formation rule is postulated, stating that $t{:}F$ is a formula whenever t is a proof polynomial and F is an arbitrary formula, hence the language of LP is a general propositional proof-carrying language. According to the completeness theorems from [3, 4], LP captures exactly the set of all valid logical principles concerning propositions and mathematical proofs with a fixed, sufficiently rich set of operations on proofs. Moreover, by the realization theorem from [3, 4], proof polynomials suffice to recover the explicit provability content in all S4-theorems (and therefore all

[2] All these systems of the logic of proofs lie within Gabbay's Labelled Deductive System framework [21].

intuitionistic propositional theorems) by realizing modalities in the latter with appropriate proof terms. In a more general setting, LP may be regarded as a device that makes reasoning about knowledge explicit and keeps track of the justification.

Here are some formal definitions.

Definition 1. *Proof polynomials* are terms built from *proof variables* $x, y, z, \ldots$ and *proof constants* $a, b, c, \ldots$ by means of three operations: *application* "$\cdot$" (binary), *union* "$+$" (binary), and *proof checker* "!" (unary).

Definition 2. Using t to stand for any proof polynomial and S for any sentence variable, the formulas are defined by the grammar

$$A = \bot \mid S \mid A_1 \to A_2 \mid A_1 \wedge A_2 \mid A_1 \vee A_2 \mid \neg A \mid t{:}A \ .$$

We assume that "$t{:}(\ )$" and "$\neg$" bind stronger than "$\wedge$" and "$\vee$," which bind stronger than "$\to$."

Definition 3. The *logic of proofs* LP has the following Hilbert-style axioms and rules:

I. The standard set of axioms for classical propositional logic, for example, A1-A10 from [28]
R1. *Modus Ponens*

II. LP1. $s{:}(F \to G) \ \to \ (t{:}F \to (s \cdot t){:}G)$ (*application*)
LP2. $t{:}F \ \to !t{:}(t{:}F)$ (*proof checker*)
LP3. $s{:}F \to (s+t){:}F, \quad t{:}F \to (s+t){:}F$ (*union*)
LP4. $t{:}F \to F$ (*reflexivity*)
R2. $\vdash c{:}A$, where A is an axiom from I - II and c is a proof constant (*constant specification rule*)

The principle LP1 specifies the basic operation of application: a justification of an implication $F \to G$ applied to any justification of the premise F returns a justification of the conclusion G. LP2 is the verifiability property of evidence: for any evidence t of F, the result of applying a checker to t, $!t$, provides a justification of $t{:}F$. LP3 reflects the monotonicity principle: a justification for F remains a justification after adding any additional evidence. Finally, LP4 is the reflexivity property.

A *constant specification* $\mathcal{CS}$ is a set $\{c_1{:}A_1, c_2{:}A_2, \ldots\}$ of formulas in which each A_i is an axiom from I-II and each c_i is a proof constant. By default, with each derivation in LP we associate a constant specification $\mathcal{CS}$ that consists of formulas introduced in this derivation by the rule of *constant specification*. The claim that F is derivable in LP is equivalent to the existence of a derivation with a constant specification $\mathcal{CS}$ associated with this derivation, i.e:

$$F \textit{ is derivable given } c_1{:}A_1, \ldots, c_n{:}A_n \ .$$

LP is closed under substitutions of proof polynomials for proof variables and formulas for propositional variables, and LP enjoys the deduction theorem.

In addition to the arithmetical completeness theorem, LP enjoys two fundamental properties: *internalization* and *realizability*.

Proposition 1. (Internalization) *If* $A_1, \ldots, A_k \vdash F$ *then for some proof polynomial* $p(x_1, \ldots, x_k)$

$$x_1{:}A_1, \ldots, x_k{:}A_k \vdash p(x_1, \ldots, x_k){:}F \ .$$

Proposition 2. (Realizability) *There is an effective procedure that constructs a realization r, which substitutes proof polynomials for all modalities in a given* S4*-derivation of formula F and thereby produces formula F^r derivable in* LP.

The logic of proofs LP may be regarded as the explicit version of S4. A paper [8] introduced a variant of the logic of proofs corresponding to S5. Logics of proofs corresponding to the modal logics K, K4, D, D4, and T were described in [10, 11].

4 Basic epistemic logic with justification

We introduce the basic epistemic logic with justifications, S4LP, consisting of S4 as the "knowledge component" and LP as the "justification component" together with the principle $t{:}F \to \Box F$ connecting justification with knowledge.

Definition 4. *Proof polynomials* for S4LP are the same as proof polynomials for LP, i.e., they are terms built from *variables* $x, y, z, \ldots$ and *constants* $a, b, c, \ldots$ by means of three operations, *application* "$\cdot$" (binary), *union* "$+$" (binary), and *evidence checker* "!" (unary). Formulas of the language of S4LP are defined by the grammar

$$A = \bot \mid S \mid A_1 \to A_2 \mid A_1 \wedge A_2 \mid A_1 \vee A_2 \mid \neg A \mid \Box A \mid t{:}A \ .$$

We assume also that "$\Box$," binds stronger than "$\wedge$" and "$\vee$."

Definition 5. The system S4LP has the following axioms and rules:

I. **Classical propositional logic**

The standard set of axioms, for example, A1-A10 from [28]
R1. *Modus ponens*

II. **Logic of Proofs** LP

LP1. $s{:}(F \to G) \ \to \ (t{:}F \to (s \cdot t){:}G)$ (*application*)
LP2. $t{:}F \ \to !t{:}(t{:}F)$ (*inspection*)
LP3. $s{:}F \to (s+t){:}F, \quad t{:}F \to (s+t){:}F$ (*union*)
LP4. $t{:}F \to F$ (*reflexivity*)
R2. $\vdash c{:}A$, where A is an axiom from I-IV and c is a proof constant (*constant specification*)

III. **Basic Epistemic Logic** S4

E1. $\Box(F \to G) \to (\Box F \to \Box G)$
E2. $\Box F \to \Box\Box F$
E3. $\Box F \to F$
R3. $\vdash F \ \Rightarrow \vdash \Box F$

IV. **Principle connecting explicit and implicit knowledge**

C1. $t{:}F \to \Box F$ (*justification-knowledge connection*)

Obviously, $\mathsf{S4LP}$ contains both LP and $\mathsf{S4}$. The principle LP4 is redundant, but we keep it listed for convenience. $\mathsf{S4LP}$ is closed under substitutions of proof polynomials for proof variables and formulas for sentence variables. $\mathsf{S4LP}$ also enjoys the deduction theorem.

Consider a *constant specification* $\mathcal{CS} = \{c_1{:}A_1, c_2{:}A_2, \ldots\}$ (where each A_i is an axiom from I-IV and each c_i is a proof constant). By $\mathsf{S4LP}_{CS}$ we mean a subsystem of $\mathsf{S4LP}$ where R2 is restricted to producing formulas from a given $\mathcal{CS}$ only. In particular, $\mathsf{S4LP}_\emptyset$ is the subsystem of $\mathsf{S4LP}$ without R2.

Lemma 1. *The principle of positive introspection*

$$t{:}F \to \Box t{:}F$$

is provable in $\mathsf{S4LP}_\emptyset$ *(hence in* $\mathsf{S4LP}_{CS}$ *for any constant specification* $\mathcal{CS}$*).*

Proof.

$t{:}F \to !t{:}(t{:}F)$	by LP2
$!t{:}(t{:}F) \to \Box t{:}F$	by C1
$t{:}F \to \Box t{:}F$	by propositional logic

$\Box$

Lemma 2. $\mathsf{S4LP}_{CS} \vdash F \quad \Leftrightarrow \quad \mathsf{S4LP}_\emptyset \vdash \bigwedge \mathcal{CS} \to F.$

Proof. The direction "$\Leftarrow$" is straightforward. "$\Rightarrow$" is proved by induction on the derivation of F in $\mathsf{S4LP}_{CS}$. The only interesting case is the rule of necessitation R3. If F is obtained by the necessitation rule R3, i.e., F is $\Box G$ and $\mathsf{S4LP}_{CS} \vdash G$, then by the induction hypothesis, $\mathsf{S4LP}_\emptyset \vdash \bigwedge \mathcal{CS} \to G$. By $\mathsf{S4}$ reasoning,

$$\mathsf{S4LP}_\emptyset \vdash \Box \bigwedge \mathcal{CS} \to \Box G \ .$$

By positive introspection (Lemma 1) and some trivial $\mathsf{S4}$ reasoning,

$$\mathsf{S4LP}_\emptyset \vdash \bigwedge \mathcal{CS} \to \Box \bigwedge \mathcal{CS} \ ,$$

hence $\mathsf{S4LP}_\emptyset \vdash \bigwedge \mathcal{CS} \to F$. $\Box$

Lemma 3. *For any formula F, there are proof polynomials* $\mathtt{up}_F(x)$ *and* $\mathtt{down}_F(x)$ *such that* $\mathsf{S4LP}$ *proves*

1. $x{:}F \to \mathtt{up}_F(x){:}\Box F$
2. $x{:}\Box F \to \mathtt{down}_F(x){:}F$.

Proof.

1.	$x{:}F \to \Box F$	by C1
	$a{:}(x{:}F \to \Box F)$	specifying constant a, by R2
	$!x{:}(x{:}F) \to (a{\cdot}!x){:}\Box F$	by LP1 and propositional logic
	$x{:}F \to !x{:}(x{:}F)$	by LP2
	$x{:}F \to (a{\cdot}!x){:}\Box F$	by propositional logic

It suffices now to set $\mathtt{up}_F(x)$ to $a{\cdot}!x$ with $a{:}(x{:}F \to \Box F)$.

2.	$\Box F \to F$	by E3
	$b{:}(\Box F \to F)$	specifying constant b, by R2
	$x{:}\Box F \to (b \cdot x){:}F$	by LP1 and propositional logic

It suffices now to set $\mathtt{down}_F(x)$ to $b \cdot x$ with $b{:}(\Box F \to F)$. □

Proposition 3. (Constructive necessitation) *If* $\mathsf{S4LP} \vdash F$*, then* $\mathsf{S4LP} \vdash p{:}F$ *for some proof polynomial* p*.*

Proof. Induction on a derivation of F. Base: F is an axiom. Then use constant specification rule. In this case, p is an arbitrary proof constant and $p{:}F$ is included in the constant specification corresponding to this derivation. Induction step: Let F be obtained from $X \to F$ and X by *modus ponens*. By the induction hypothesis, $\vdash s{:}(X \to F)$ and $\vdash t{:}X$, hence by LP1, $\vdash (s \cdot t){:}F$ and hence p is $s \cdot t$. If F is obtained by R2, then F is $c{:}A$ for some constant c and axiom A. Use the axiom LP2 to derive $!c{:}(c{:}A)$, i.e., $!c{:}F$. Here p is $!c$. If F is obtained by R3, then $F = \Box G$ and $\vdash G$. By the induction hypothesis, $\vdash t{:}G$ for some proof polynomial t. Use Lemma 3.1 to conclude that $\vdash \mathtt{up}_G(t){:}\Box G$, and put $p = \mathtt{up}_G(t)$.

Note that the proof polynomial p is always a ground term built from proof constants by applications and proof checker operations only. Moreover, the presented derivation of $p{:}F$ does not use rule R3. □

The necessitation rule R3 is derivable from the rest of $\mathsf{S4LP}$. Indeed, if $\vdash F$ then, by Proposition 3, $\vdash p{:}F$ for some proof polynomial p. By C1, $\vdash \Box F$. However, the rule of necessitation is not redundant in $\mathsf{S4LP}_{CS}$ for finite constant specifications. To emulate the rule of necessitation one needs to apply constructive necessitation to the unbounded set of theorems of $\mathsf{S4LP}_{CS}$, which requires an unbounded set of constant specifications.

The following property of $\mathsf{S4LP}$ is a generalization of constructive necessitation (Proposition 3). It is the explicit analogue of the rule

$$\frac{A_1, \ldots, A_k, \Box B_1, \ldots, \Box B_n \vdash F}{\Box A_1, \ldots, \Box A_k, \Box B_1, \ldots, \Box B_n \vdash \Box F}$$

which holds in any normal modal logic containing $\mathsf{K4}$.

Proposition 4. (Lifting) *If* $A_1, \ldots, A_k, y_1 : B_1, \ldots, y_n : B_n \vdash F$*, then for some proof polynomial* $p(x_1, \ldots, x_k, y_1, \ldots, y_n)$

$$x_1{:}A_1, \ldots, x_k{:}A_k, y_1{:}B_1, \ldots, y_n{:}B_n \vdash p(x_1, \ldots, x_k, y_1, \ldots, y_n){:}F \ .$$

Proof. Similar to Proposition 3 with two new base clauses. If F is A_i, then x_i can be taken as p. If F is $y_j{:}B_j$, then p is equal to $!y_j$. □

Proposition 5. (Internalization) *If* $A_1, \ldots, A_k \vdash F$*, then for some proof polynomial* $p(x_1, \ldots, x_k)$

$$x_1{:}A_1, \ldots, x_k{:}A_k \vdash p(x_1, \ldots, x_k){:}F.$$

Proof. A special case of Proposition 4. □

The internalization property states that any derivation in $\mathsf{S4LP}$ can be internalized as a proof polynomial and verified in $\mathsf{S4LP}$ itself.

Note that axiom C1 in $\mathsf{S4LP}$ can be replaced by the *explicit positive introspection* principle $t{:}F \to \Box t{:}F$. The new system will coincide with $\mathsf{S4LP}$ modulo replacement of some constants by ground proof polynomials.

5 Introducing explicit negative introspection

As was noticed earlier, the *explicit negative introspection* principle

$$\neg t{:}F \to \Box\neg t{:}F$$

holds when we interpret $\Box$ as mathematical provability, thus suggesting it as an epistemic principle.

Definition 6. The system $\mathsf{S4LPN}$ has the same syntax, axioms, and rules as $\mathsf{S4LP}$ with one additional axiom:

C2. $\neg t{:}F \to \Box\neg t{:}F$ (*explicit negative introspection*)

$\mathsf{S4LPN}_{\mathcal{CS}}$ is $\mathsf{S4LPN}$ with the rule R2 limited to a given constant specification $\mathcal{CS}$. $\mathsf{S4LPN}_{\emptyset}$ is $\mathsf{S4LPN}_{\mathcal{CS}}$ with the empty constant specification.

Analogues of Lemmas 1, 2, and 3 as well as Propositions 3, 4, and 5 hold for $\mathsf{S4LPN}$ as well.

Lemma 4. *The principle of decidability of evidence*

$$\Box t{:}F \vee \Box\neg t{:}F$$

is provable in $\mathsf{S4LPN}$ *(hence in* $\mathsf{S4LPN}_{\mathcal{CS}}$ *for any constant specification* $\mathcal{CS}$*).*

Proof.

$t{:}F \to \Box t{:}F$	positive introspection
$\neg t{:}F \to \Box\neg t{:}F$	by negative introspection
$(t{:}F \vee \neg t{:}F) \to (\Box t{:}F \vee \Box\neg t{:}F)$	by propositional logic
$\Box t{:}F \vee \Box\neg t{:}F$	by propositional logic

$\Box$

Alternatively, $\mathsf{S4LPN}$ can be axiomatized over $\mathsf{S4LP}$ by the principle of decidability of evidence modulo replacement of some constants by ground proof polynomials.

6 Models

We work with semantics that uses the idea, which can be traced back to Mkrtychev [35] and Fitting ([17]), of augmenting Boolean or Kripke models with an evidence function, which assigns "admissible evidence" terms to a statement. The statement $t{:}\varphi$ holds in a given world u iff both of the following conditions are met:

1) t is an admissible evidence for φ in u;

2) φ holds in all worlds accessible from u.

One more idea came from the paper [6] which suggested taking an "evidence accessibility" relation different from the knowledge accessibility relation, thus semantically separating explicit knowledge from usual knowledge.

A *frame* is a structure (W, R, R^e), where W is a non-empty set of *states* (*possible worlds*), R is a binary *accessibility* relation on W, and R^e is a binary *evidence accessibility* relation on W. For our purposes, the relations R and R^e can be taken as reflexive and transitive. R^e should contain but not necessarily coincide with R.

Given a frame (W, R, R^e), a *possible evidence* function $\mathcal{E}$ is a mapping from worlds and justification terms to sets of formulas. We can read $F \in \mathcal{E}(u, t)$ as "F is one of the formulas for which t serves as possible evidence in world u." An evidence function must obey conditions that respect the intended meanings of the operations on justification terms (i.e., proof polynomials).

Definition 7. $\mathcal{E}$ is an evidence function on (W, R, R^e) if for all proof polynomials s and t, for all formulas F and G, and for all $u, v \in W$, each of the following hold:

1. *Monotonicity*: uR^ev implies $\mathcal{E}(u,t) \subseteq \mathcal{E}(v,t)$.
2. *Application*: $F \to G \in \mathcal{E}(u,s)$ and $F \in \mathcal{E}(u,t)$ implies $G \in \mathcal{E}(u, s{\cdot}t)$.
3. *Inspection*: $F \in \mathcal{E}(u,t)$ implies $t{:}F \in \mathcal{E}(u, !t)$.
4. *Sum*: $\mathcal{E}(u,s) \cup \mathcal{E}(u,t) \subseteq \mathcal{E}(u, s+t)$.

A model is a structure $\mathcal{M} = (W, R, R^e, \mathcal{E}, \Vdash)$ where (W, R, R^e) is a frame with an evidence function $\mathcal{E}$ on (W, R, R^e) and $\Vdash$ is an arbitrary mapping from sentence variables to subsets of W.

Given a model $\mathcal{M} = (W, R, R^e, \mathcal{E}, \Vdash)$, the forcing relation $\Vdash$ is extended from sentence variables to all formulas by the following rules. For each $u \in W$:

1. $\Vdash$ respects Boolean connectives ($u \Vdash F \wedge G$ iff $u \Vdash F$ and $u \Vdash G$, $u \Vdash \neg F$ iff $u \nVdash F$, etc.).
2. $u \Vdash \Box F$ iff $v \Vdash F$ for every $v \in W$ with uRv.
3. $u \Vdash t{:}F$ iff $F \in \mathcal{E}(u,t)$ and $v \Vdash F$ for every $v \in W$ with uR^ev.

We say F is *true* at a world $u \in W$ if $u \Vdash F$; otherwise, F is *false* at u. Informally speaking, $t{:}F$ is true in a given world u iff t is an acceptable evidence term for F in u and F is true in all worlds v accessible from u via the evidence accessibility relation R^e. A formula F is *true* in a model if F is true at each world of the model; F is valid if F is true in every model. Given a constant specification $\mathcal{CS}$, a model $\mathcal{M}$ *meets* $\mathcal{CS}$ if $\mathcal{M} \Vdash c{:}A$ whenever $c{:}A \in \mathcal{CS}$.

The following lemma is a straightforward corollary of the definitions:

Lemma 5. *$u \Vdash t{:}F$ and uR^ev yield $v \Vdash t{:}F$.*

The above models with singleton W's are called *Mkrtychev models* (M-*models*, for short). M-models were introduced in [35] under the name of pre-models. The logic of proofs LP was shown in [35] to be sound and complete with respect to M-models.

We call models with $R = R^e$ *Fitting models* (F-*models*). They were first introduced in [17] under the name *weak models* as an epistemic semantics for the logic of proofs LP. In [9, 18], it was shown that F-models work for S4LP as well.

Finally, we call arbitrary models of the above class AF-*models*. AF-models were introduced in [6] in a general setting for several agents where the need to separate knowledge and explicit knowledge became apparent. AF-models work for a wide class of systems, including the ones mentioned above (LP, S4LP, and S4LPN).

Theorem 1. *For any given constant specification $\mathcal{CS}$, the logic $\mathsf{S4LP}_{CS}$ is sound and complete with respect to AF-models that meet $\mathcal{CS}$.*

Proof. Soundness is straightforward. S4-axioms and rules hold because an AF-model with respect to the modal language is the usual Kripke model for S4. LP axioms and rules are guaranteed by the properties of the evidence function $\mathcal{E}$. Let us check the connection axiom $t{:}F \to \Box F$. Suppose $u \Vdash t{:}F$ and uRv. Then uR^ev, since $R \subseteq R^e$, and $v \Vdash F$. Hence, $u \Vdash \Box F$.

Completeness is established by the standard maximal consistent set construction. First of all, we define the canonical model $(W, R, \mathcal{E}, \Vdash)$ for $\mathsf{S4LP}_{CS}$. Call a set S of formulas in the language of

$\mathsf{S4LP}_{CS}$ *consistent* if for no $F_1, \ldots, F_n \in S$, $\neg(F_1 \wedge \ldots \wedge F_n)$ is provable in $\mathsf{S4LP}_{CS}$. Consistent sets extend to *maximal consistent* sets by the standard Lindenbaum construction. W is the collection of all maximal consistent sets. Define $\Gamma^\sharp$ as $\{\Box F \mid \Box F \in \Gamma\}$ and $\Gamma^\flat$ as $\{t{:}F \mid t{:}F \in \Gamma\}$. The accessibility relation R, the evidence accessibility relation R^e and the evidence function $\mathcal{E}$ are defined by

$$\Gamma R \Delta \qquad \text{iff} \qquad \Gamma^\sharp \subseteq \Delta \ ,$$

$$\Gamma R^e \Delta \qquad \text{iff} \qquad \Gamma^\flat \subseteq \Delta \ ,$$

and

$$F \in \mathcal{E}(\Gamma, t) \qquad \text{iff} \qquad t{:}F \in \Gamma \ .$$

Obviously, R^e extends R. Indeed, let $\Gamma R \Delta$ and $t{:}F \in \Gamma$ hold. Then $\Box t{:}F \in \Gamma$, since $\mathsf{S4LP}_{CS} \vdash t{:}F \to \Box t{:}F$. By $\Gamma R \Delta$, we conclude that $\Box t{:}F \in \Delta$. Since $\mathsf{S4LP}_{CS} \vdash \Box t{:}F \to t{:}F$, $t{:}F \in \Delta$ as well. So, $R \subseteq R^e$. Hence, (W, R, R^e) is an $\mathsf{S4LP}$-frame.

Let us check the evidence function properties.

Monotonicity: $F \in \mathcal{E}(\Gamma, t)$ yields $t : F \in \Gamma$. If $\Gamma R^e \Delta$, then $t : F \in \Delta$, by the definition of R^e. By the definition of $\mathcal{E}$, $F \in \mathcal{E}(\Delta, t)$. *Application*: $F \to G \in \mathcal{E}(\Gamma, s)$ and $F \in \mathcal{E}(G, t)$ implies $s{:}(F \to G) \in \Gamma$ and $t{:}F \in \Gamma$. Since $s{:}(F \to G) \to (t{:}F \to (s \cdot t){:}G) \in \Gamma$ and Γ is closed under *modus ponens* (as a maximal consistent set of formulas), $(s \cdot t){:}G \in \Gamma$. Hence, $G \in \mathcal{E}(u, s \cdot t)$. A similar argument proves *inspection* and *sum*.

Finally, for each propositional letter p, define

$$\Gamma \Vdash p \qquad \text{iff} \qquad p \in \Gamma \ .$$

Lemma 6. (Truth Lemma) *For each formula F and each $\Gamma \in W$,*

$$\Gamma \Vdash F \qquad \textit{iff} \qquad F \in \Gamma \ .$$

Proof. Induction on F. The base case is given by the definitions and the cases of Boolean connectives are standard.

Case: F is $\Box X$.

If $\Box X \in \Gamma$, then $\Box X \in \Delta$ for each Δ such that $\Gamma R \Delta$. Since $\mathsf{S4LP}_{CS} \vdash \Box X \to X$, $X \in \Delta$. By the induction hypothesis, $\Delta \Vdash X$, hence, $\Gamma \Vdash \Box X$.

If $\Box X \notin \Gamma$, then $\Gamma^\sharp \cup \{\neg X\}$ is a consistent set. If it were not consistent, then $\mathsf{S4LP}_{CS} \vdash \Box Y_1 \wedge \Box Y_2 \wedge \ldots \wedge \Box Y_n \to X$ for some $\Box Y_1, \Box Y_2, \ldots, \Box Y_n \in \Gamma$. By $\mathsf{S4}$ reasoning, $\mathsf{S4LP}_{CS} \vdash \Box Y_1 \wedge \Box Y_2 \wedge \ldots \wedge \Box Y_n \to \Box X$, hence $\Box X \in \Gamma$, a contradiction. So, $\Gamma^\sharp \cup \{\neg X\}$ is consistent. Take Δ to be a maximal consistent extension of $\Gamma^\sharp \cup \{\neg X\}$. It is apparent that $\Delta \in W$, $\Gamma R \Delta$ and $X \notin \Delta$. By the definition of a model, $\Delta \nVdash X$, hence $\Gamma \nVdash \Box X$.

Case: F is $t{:}X$.

Let $t{:}X \in \Gamma$. Then $X \in \mathcal{E}(\Gamma, t)$. By the definition of R^e, $t{:}X \in \Delta$ for each Δ such that $\Gamma R^e \Delta$. Since $\mathsf{S4LP}_{CS} \vdash t{:}X \to X$, $X \in \Delta$ as well. By the induction hypothesis, $\Delta \Vdash X$. By the definition of forcing at node Γ, $\Gamma \Vdash t{:}X$.

If $\Gamma \Vdash t{:}X$, then $X \in \mathcal{E}(\Gamma, t)$, hence $t{:}X \in \Gamma$, by the definition of $\mathcal{E}$. $\Box$

To conclude the proof of Theorem 1, suppose $\mathsf{S4LP}_{CS} \nvdash F$. Then $\{\neg F\}$ is a consistent set. Take its maximal consistent extension Γ. Then $F \notin \Gamma$ and, by the Truth Lemma, $\Gamma \nVdash F$ in the canonical model. $\Box$

Comment 1. The completeness of $\mathsf{S4LP}$ with respect to F-models was proved in [18], where the canonical model W, R, $\mathcal{E}$ and $\Vdash$ were chosen as above and R^e was defined as $R^e = R$. The completeness proof there is essentially the same as in Theorem 1 with the following two minor deviations.

1. To establish the *monotonocity* property of $\mathcal{E}$, assume $F \in \mathcal{E}(\Gamma, t)$. Then $t{:}F \in \Gamma$ and $\Box t{:}F \in \Gamma$ by positive introspection in $\mathsf{S4LP}_{CS}$. By the definition of R^e as R, $\Box t{:}F \in \Delta$ for each Δ such that $\Gamma R^e \Delta$. By reflexivity, $t{:}F \in \Delta$. Hence, $F \in \mathcal{E}(\Delta, t)$.

2. The case $t{:}X \in G$ in the Truth Lemma. By the definition of $\mathcal{E}$, $X \in \mathcal{E}(\Gamma, t)$. Take Δ such that $\Gamma R^e \Delta$, i.e., $\Gamma R \Delta$. By positive introspection, $\Box t{:}X \in \Gamma$, hence $\Box t{:}X \in \Delta$. By reflexivity, $t{:}X \in \Delta$ and $X \in \Delta$. By the induction hypothesis, $\Delta \Vdash X$. By the definition of forcing, $\Gamma \Vdash t{:}X$.

Theorem 2. *For any constant specification $\mathcal{CS}$, $\mathsf{S4LPN}_{CS}$ is sound and complete with respect to AF-models with symmetric R^e meeting $\mathcal{CS}$.*

Proof. Let $(W, R, R^e, \mathcal{E}, \Vdash)$ be an AF-model from the formulation of the theorem. By the definitions, $R \subseteq R^e$, R is reflexive and transitive, whereas R^e is reflexive, symmetric, and transitive, i.e., R^e is an equivalence relation on W that extends R.

The soundness portion can be established by a straightforward induction on derivations in $\mathsf{S4LPN}_{CS}$. All the cases but C2 follow from AF-soundness of $\mathsf{S4LP}$, cf. Theorem 1. Let us check C2. Suppose $u \Vdash \neg t{:}F$, and pick v such that uRv. Suppose $v \nVdash \neg t{:}F$, i.e., $v \Vdash t{:}F$. Since vR^eu, by Lemma 5, $u \Vdash t{:}F$ – a contradiction. Actually, we have shown the stability property of AF-models of the above kind: each formula $t{:}F$ either holds at all worlds of a given equivalence class with respect to R^e, or it does not hold in all worlds of this class.

The completeness portion is proved by the maximal consistent sets construction. Define the canonical model for $\mathsf{S4LPN}_{CS}$. Call a set S of formulas in the language of $\mathsf{S4LPN}_{CS}$ *consistent* if for no $F_1, \ldots, F_n \in S$, is $\neg(F_1 \wedge \ldots \wedge F_n)$ provable in $\mathsf{S4LPN}_{CS}$. Consistent sets extend to *maximal consistent* sets by the Lindenbaum construction. W is the collection of all maximal consistent sets. As before, $\Gamma^\sharp = \{\Box F \mid \Box F \in \Gamma\}$ and $\Gamma^\flat = \{t{:}F \mid t{:}F \in \Gamma\}$. Define R, R^e, and $\mathcal{E}$ by

$$\Gamma R \Delta \qquad \text{iff} \qquad \Gamma^\sharp \subseteq \Delta \ ,$$

$$\Gamma R^e \Delta \qquad \text{iff} \qquad \Gamma^\flat = \Delta^\flat \ ,$$

and

$$F \in \mathcal{E}(\Gamma, t) \qquad \text{iff} \qquad t{:}F \in \Gamma \ .$$

Finally, for each propositional letter p, define

$$\Gamma \Vdash p \qquad \text{iff} \qquad p \in \Gamma \ .$$

Let us check that (W, R, R^e) is an $\mathsf{S4LPN}$-frame. Clearly, R is reflexive and transitive, and R^e is an equivalence relation. Furthermore, $R \subseteq R^e$. Indeed, let $\Gamma R \Delta$ and $t{:}F \in \Gamma$. Since $\mathsf{S4LPN}_{CS} \vdash t{:}F \to \Box t{:}F$, the latter formula is in Γ, hence $\Box t{:}F \in \Gamma$ as well. Since $\Gamma R \Delta$, $\Box t{:}F \in \Delta$. Since $\mathsf{S4LPN}_{CS} \vdash \Box t{:}F \to t{:}F$, $t{:}F \in \Delta$. So, $\Gamma^\flat \subseteq \Delta^\flat$. Let $t{:}F \notin \Gamma$. By maximality, $\neg t{:}F \in \Gamma$. Since $\mathsf{S4LPN}_{CS} \vdash \neg t{:}F \to \Box \neg t{:}F$, the latter formula is in Γ, hence $\Box \neg t{:}F \in \Gamma$ as well. Since $\Gamma R \Delta$, $\Box \neg t{:}F \in \Delta$, hence $\neg t{:}F \in \Delta$ and $t{:}F \notin \Delta$. Therefore, $\Gamma^\flat \supseteq \Delta^\flat$ and $\Gamma^\flat = \Delta^\flat$.

Let us check the properties of the evidence function.

Monotonicity: Let $F \in \mathcal{E}(\Gamma, t)$ and $\Gamma R^e \Delta$. By the definition of $\mathcal{E}(\Gamma, t)$, $t{:}F \in \Gamma$. Hence $t{:}F \in \Gamma$, since $G^\flat = \Delta^\flat$. So, $F \in \mathcal{E}(\Gamma, t)$. *Application*, *inspection*, and *sum* follow immediately from the definitions.

Lemma 7. (Truth Lemma) *For each formula F,*

$$\Gamma \Vdash F \text{ iff } F \in \Gamma \ .$$

Proof. Induction on F. The base case is given by the definitions and the cases of Boolean connectives are standard.

Case: F is $\Box X$ is treated similarly to Lemma 6.

Case: F is $t{:}X$.

If $t{:}X \in \Gamma$, then $X \in \mathcal{E}(\Gamma, t)$. Let $\Gamma R^e \Delta$. By the definition of R^e, $t{:}X \in \Delta$. Since $\mathsf{S4LPN}_{CS} \vdash t{:}X \to X$, $X \in \Delta$. So, by the induction hypothesis, $\Delta \Vdash X$. By the definitions, $\Gamma \Vdash t{:}X$.

If $\Gamma \Vdash t{:}X$, then $X \in \mathcal{E}(\Gamma, t)$, hence $t{:}X \in \Gamma$, by the definition of $\mathcal{E}$. $\Box$

A standard argument concludes the proof of the theorem. Suppose $\mathsf{S4LPN}_{CS} \nvdash F$. Then the set $\{\neg F\}$ is consistent and let Γ be its maximal consistent extension. Then $F \notin \Gamma$ and, by Lemma 7, $\Gamma \nVdash F$. $\Box$

The following stronger form of the completeness theorem holds:

Theorem 3. *For each F such that $\mathsf{S4LPN}_{CS} \nvdash F$, there is an AF-model $\widehat{M}$ that meets $\mathcal{CS}$ such that the evidence accessibility relation in $\widehat{M}$ is total and F is false in $\widehat{M}$.*

Proof. Take the canonical model for $\mathsf{S4LPN}_{CS}$ and a world Γ_0 such that $\Gamma_0 \nVdash F$. Consider the equivalence class $\widehat{W}$ with respect to R^e such that $\Gamma_0 \in \widehat{W}$. Since $R \subseteq R^e$, $\widehat{W}$ is closed under accessibility relation R: if $\Gamma \in \widehat{W}$ and $\Gamma R \Delta$, then $\Delta \in \widehat{W}$. Let $\widehat{R}$, $\widehat{R^e}$, $\widehat{\mathcal{E}}$, and $\widehat{\Vdash}$ be R, R^e, $\mathcal{E}$, and $\Vdash$ restricted to $\widehat{W}$, respectively. The resulting structure is an AF-model

$$\widehat{M} = (\widehat{W}, \widehat{R}, \widehat{R^e}, \widehat{\mathcal{E}}, \widehat{\Vdash})$$

with the evidence accessibility relation $\widehat{R^e}$ total on its domain $\widehat{W}$. Indeed, we have already checked that $(\widehat{W}, \widehat{R}, \widehat{R^e})$ is an AF-frame. The properties of the evidence function $\widehat{\mathcal{E}}$ are nothing but the special cases of the corresponding properties for $\mathcal{E}$. Furthermore, for each formula X and each $\Gamma \in \widehat{W}$,

$$\Gamma \Vdash X \quad \text{iff} \quad \Gamma \,\widehat{\Vdash}\, X \ ,$$

since $\widehat{W}$ is closed with respect to both R and R^e.

Since $\Gamma_0 \nVdash F$, $\Gamma_0 \ \widehat{\nVdash} F$ as well. This concludes the proof of Theorem 3. $\Box$

7 Arithmetical semantics for S4LP and S4LPN

Arithmetical semantics for S4LP and S4LPN is given by interpreting $\Box F$ via the strong provability operator

F is true and provable in Peano Arithmetic PA,

together with interpreting $t{:}F$ as before:

t is a proof of F in Peano Arithmetic PA.

Using the strong provability operator to obtain S4-compliant logics has been a well-established tradition in provability logic (cf. [1, 13, 14, 25, 30, 31, 37, 38]).

Theorem 4. (Arithmetical soundness of epistemic logic with justification) *Let $\mathcal{CS}$ be a finite constant specification. If* $\mathsf{S4LPN}_{CS} \vdash F$*, then F is true under any arithmetical interpretation which translates $\Box F$ as strong provability in* PA *and $t{:}F$ as "t is a proof of F in* PA*."*

Proof. By induction on F. The validity of LP axioms and rules was shown in [3, 4]. The validity of S4 axioms and rules under the strong provability interpretation was shown in many sources, cf. [14].

The validity of the connection axiom $t{:}F \to \Box F$ is a combination of the validity of the explicit reflection $t{:}F \to F$, which is an LP axiom, already checked, and a first order tautology $Prf(t, \varphi) \to \exists x Prf(x, \varphi)$, where $Prf(x, y)$ is an arithmetical formula for *x is a proof of y*.

Finally, the negative introspection axiom $\neg t{:}F \to \Box \neg t{:}F$ is a special case of σ-completeness of the arithmetic, cf. [14]. $\Box$

An arithmetically complete system GrzLPN of strong provability with proofs can be axiomatized by adding to S4LP the modal axiom by Grzegorczyk $\Box(\Box(F \to \Box F) \to F) \to F$. Models for GrzLPN are F-models with reflexive partially ordered frames. This can be established by a combination of the methods from [9, 37, 38].

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