Strategic Communication

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ABSTRACT

We model games where players strategically exchange messages in a language for reasoning and strategically update their reasoning. The language for the stage game incorporates awareness and knowledge and extends [14]'s propositional quantification to quantification over all sentences in the language. The updating of reasoning is modeled as a strategic choice of the players and the dynamics of the logic provide constraints for this strategic update choice. A communication game is constructed using an underlying incomplete information game, the strategic choice of messages and the strategic and logic dynamics. Multiple games are described varying by how the game theoretic type-space relates to the language for reasoning.

Categories and Subject Descriptors

F.4.1 [**Mathematical Logic**]: Modal Logic; F.4.1 [**Mathematical Logic**]: Proof Theory; I.2.3 [**Artificial Intelligence**]: Deduction and Theorem Proving

General Terms

Economics, Theory

Keywords

Game Theory, Unawareness, Communication

1. INTRODUCTION

Communication in the form of the exchange of messages is among the most prevalent of human activities. From the physician to the corporate executive, individuals spend much of their day sending, receiving and processing messages, messages that ultimately affect more substantial behavior. The exercise of writing of this paper involves the author phrasing and ordering messages that try to convince you to keep reading and to judge kindly this piece of work as well as the author. And whether in a committee meeting, a paper, or a casual conversation, communication takes place within a broader context of individuals' preferences and processes of reasoning. My objective, in this paper, is to model the strategic exchange of messages.

I begin with a language, which represents the information and views the players have about the state of the world. It is natural to assume that players can say what they think, and vice versa, hence the set of messages that might be communicated will be sentences in the language. While the players may communicate what they know, or believe, it seems that in many practical cases the players may communicate aspects of the state of the world that other players did not consider. Hence, we ask that the awareness of the players be part of the events considered. In particular, players may reason about the unawareness of others. Reasoning about the unawareness of others may lead a player to reason about their own unawareness, especially if we wish the player to expect to receive messages she was unaware of. Hence, we call for a language that also allows a player to reason about the extent of her own awareness.

Once the language is set, we turn to strategic interaction. This involves the choice of message – the player's action, how a message impacts the reasoning of other players and the message choices they might make – the dynamics of reasoning. Finally, payoffs are associated with the sequence of messages sent and, most importantly, the concluding state of reasoning in which we find the players. For example, consider two players, Alice and Bob. Alice knows that a holds and can send the message "a is true." Bob knows $a \longrightarrow b$ holds. Assume the payoffs depend on whether the players know b, know $\neg b$, Alice knows one and Bob neither, or vice versa. If Alice sends the message "a is true," Bob will have to evaluate whether he should believe Alice, he needs to take into account what he thinks about a, as well as what he thinks Alice thinks about $a \longrightarrow b$, since if she is unaware of this, it might make her message more reliable. In other words, Bob needs to take into account Alice's incentives for sending this message. Hence, how Bob revises his beliefs is a strategic decision intertwined with logic dynamics. The main purpose of this paper is to disentangle the logic and strategic considerations.

A communication game has the following structure. The language we use is an extension of [14] who present a language with awareness, knowledge as well as the ability to quantify over propositional formulae, i.e. propositional quantifiers. Our language adds the ability to quantify over all sentences by adding high level quantifiers (a construction that might be of separate interest). The stage game begins with the set of sentences each player is aware of, and has one player choose a sentence to send to all other players. This choice of message is purely game theoretic. We consider a variety of incomplete information games associated with the stage game. These allow us to consider probabilistic belief and common priors, but they also allow us to connect the game theoretic type-space to a model corresponding to the logic at hand. We present ten possible constructions of incomplete information game type spaces. Once a message is chosen and sent, the game has to proceed to the next stage. Here we have the players move from one collection of sentences representing their reasoning to a new collection. We assume that this move is also game theoretic, i.e., that the players *choose* their new set of sentences. Obviously, this choice cannot be independent of the reasoning rules of the language. In fact, we may want to impose further dynamic logic restrictions on this choice. To do so, we define the logic dynamic constraints function which restricts the choices the players can make as a function of the history of reasoning and messages (or a model associated with them). We then define the game theoretic choice of reasoning as a function of the history and the player's type in the corresponding incomplete information game, such that each player is restricted to choose within the rules of the logic dynamic constraints function. At the following stage of the game we associate a new game theoretic type-space that agrees with the new state of reasoning the players have chosen. This change in type space is our one departure from standard game theoretic constructions which is needed to address the inherent unawareness in the game.

The dynamics of reasoning are separated into a logic constraint on choice and a game theoretic choice within this constraint. This separation allows us to treat the reasoning about incentives, the uncertainties about players' revision of reasoning, and the choice of reaction to messages (e.g. whether to believe), all within game theory. By considering solutions to a game, game theory does not explicitly model the process of reasoning leading to a particular behavior. Instead, it allows us to consider properties of behavior directly. In our case, this implies that we can consider when players' choice of reasoning revision corresponds to a solution without detailing what dynamic logic rules would lead to the solution. Characterizing behavior without fully detailing how people actually/should reason and make choices, is one of the attractive features of using game theory for modeling realistic strategic interactions.

The strategic communication terminates with a resulting state of reasoning for the players. It is this final state that the messages were chosen to influence. Hence, the payoffs are determined by what the players are aware of and know at the end of the communication. For example, in a coordination setting the payoff could be set as the inverse of a metric measuring the distance between players' aggregate/common knowledge (and/or awareness), or it could be the extent of agreement they achieve regarding a particular event.

The approach to payoffs is game theoretic, in the sense that there is no explicit reasoning about payoffs in the language. Similarly, there is no explicit reasoning about the strategic choice of messages or a formal reasoning process leading the player towards message selection. It is only the dynamics of reasoning that require both a logic and game theoretic treatment since it is where the two are most intertwined. On the one hand a message sent from Alice to Bob would make it common knowledge that both are aware of the message assuming we do not consider forgetfulness. However, how Bob reasons about *why* Alice might have sent the message and what deductions should he make about her knowledge, awareness and reasoning about him, will depend on Alice's incentives, i.e. on the game theoretic solution concept we use. At the same time, the solution we use must take into account that behavior should be a function of the reasoning the player *can* do. When Alice chooses a message that is supposed to anticipate the reaction of Bob and the ensuing payoffs, Alice can only make a choice based on what she is aware of (or the extent to which she reasons about her unawareness). Our solution is to take the rules imposed by logic dynamics as constraints on the game theoretic choice sets available to the players for revising their reasoning.

There is a vast literature considering strategic dynamic communication. The purely game theoretic literature usually does not treat the content of the messages (see the discussion in [9]). However, logicians have an extensive collection of related results. These can be separated to models of logic dynamics, message driven dynamics and awareness dynamics. In [13], [15], [16], [26], [28], [29] and [30] logic for awareness and dynamic awareness – becoming aware – are extensively studied in a variety of settings. In addition, the field of argumentation (and defeasible logic, see [20]) has a long tradition of discussing the dynamics of messages, as well as enlisting games to model the dynamics of sending messages as in [3], [4], [12], [22] and [23]. Argumentation theory has focused on the applicability of the reasoning (e.g. for legal deliberations, or contracting), however the game theoretic aspect is focused on the choice of a message rather than the revision of reasoning is a choice (leading mostly to a perfect information game setting).

The paper continues as follows. Section 2 presents the language for reasoning and messages which is an extension of [14] to quantification over arbitrary sentences. In section 3 we define the messages that can be sent in the game. Sections 4 defines the static game forms, i.e. a selection of incomplete information games that can be used with the communication game. Section 5 defines the logic constraints function and suggests some possible properties it might have. This is the function that restricts the game theoretic reasoning revision choices the players can make. In Section 6 the components above are brought together into game forms. We construct the explicit communication game form for three representative static game forms which vary in the degree of relation between the game theoretic representation and models in the language. Section 7 discusses solutions and in particular the notion of best response in the various communication games.

2. STATIC LANGUAGE

The language is constructed from a countable set of atomic formulae (primitive propositions) denoted $P = \{p, q, ...\}$, a disjoint countable set of variables denoted $X = \{x, y, ...\}$ and operators $\{\neg, \wedge, K_i^E, A_i, \forall^n\}$ for each player $i \in I$ and for $n = 0, 1, 2, ...$ The formulae in the language are defined inductively by finite iterations:

$$\varphi ::= \top | p | x | \neg\varphi | \varphi \wedge \varphi | K_i^E \varphi | A_i \varphi | \forall^n x \varphi$$

The *free variables* of a formula φ are defined inductively as subsets of X by $f(p) = \emptyset$, $f(x) = \{x\}$, $f(\neg\varphi) = f(\varphi)$, $f(\phi \wedge \varphi) = f(\phi) \cup f(\varphi)$, $f(K_i^E \varphi) = f(\varphi)$, $f(A_i\varphi) = f(\varphi)$

and $f(\forall^n x\varphi) = f(\varphi) \setminus \{x\}$. A *sentence* is a formula with no free variables. For every formula φ, variable x and formula ϕ, the formula $\varphi[x/\phi]$ is defined as the formula obtained by replacing all free occurrences of x in φ simultaneously by ϕ. Note that if x was the only free variable in φ and ϕ is a sentence then $\varphi[x/\phi]$ is a sentence as well. A sentence (formula) φ is said to be *a level* n sentence (formula) for $n = 0, 1, 2, ...$ if φ does not contain any operator $\forall^m$ for $m \geq n$. In particular φ is a level 0 sentence if it does not contain any quantifiers (or any variables for that matter as it is a sentence). The operators $\varphi \vee \phi, \varphi \longrightarrow \phi, \varphi \longleftrightarrow \phi, \exists^n x\varphi$ are defined as abbreviations to $\neg(\neg\varphi \wedge \neg\phi), \neg\varphi \vee \phi, (\varphi \longrightarrow \phi) \wedge (\phi \longrightarrow \varphi), \neg\forall^n x\neg\varphi$ respectively. The set of formulae in our language is denoted $\mathcal{L}$ and the set of sentences is denoted $\mathcal{S}$.

The interpretation of the statements is as follows. $A_i\varphi$ stands for "player i is aware of φ", or "i is considering φ". $K_i^E\varphi$ stands for "player i explicitly knows φ". The quantifiers $\forall^n x\varphi$ stands for "for every n level sentence ϕ we have that $\varphi[x/\phi]$ holds". The interpretation of explicit knowledge aims to capture the conjunction of the traditional knowledge operator and awareness. We follow [6] interpretation but note that adding the standard implicit knowledge operator K_i and defining $K_i^E\varphi \Longleftrightarrow K_i\varphi \wedge A_i\varphi$ is an alternative as is done in [14]. We point out that the interpretation of knowledge becomes even more delicate when one considers explicit dynamics of reasoning. In particular, [28], [29], [30] discuss how a language which allows for an operator for becoming aware interacts with explicit and implicit knowledge. We reiterate that in our setting becoming aware is treated from a game theoretic perspective and not from within the language.

Our collection of quantifiers $\forall^n$ are unique to our language and require some further discussion. As noted above, the language restricted only to the $\forall^0$ quantifier is exactly the language provided by [14]. The $\forall^0$ quantifier corresponds to a propositional. Quantifying over propositions is based on the ideas developed in [2], [10], [17] and [18] with the more recent treatment of the topic in [5]. As [14] pointed out, quantifying only over level 0 sentences guarantees that quantification is well defined since assigning for x in $\forall xx$ the formula $\forall xx$ itself, cannot be evaluated inductively. However, it might be useful to allow our players to reason about the existence of sentences that include quantifiers, in particular in a communication setting. The idea here is to allow quantification over all sentences, but instead of using a single operator that would cause a self-referential "loop", we consider a collection of quantifiers which together cover all sentences in the language (including sentences with these quantifiers themselves). In particular, for every sentence φ let $m = Max\{n | \forall^n \text{ appears in } \varphi\}$ with $m = -1$ if this set is empty. By the finiteness of sentences construction we have that m is finite hence φ is being considered in $\forall^n$ for all $n > m$.

The following example demonstrates the usefulness of including sentences that consider the existence of quantified sentences. Consider two players and the sentence

$$\varphi = \exists^0 x(K_2x \wedge \neg A_1x \wedge (x \longrightarrow a)) \vee \exists^0 x(K_2x \wedge \neg A_1x \wedge (x \longrightarrow \neg a))$$

which states that there is some sentence (that has no quantification) that player 2 knows, player 1 cannot reason about and such that it implies a holds, or that there is a sentence of the same sort that implies $\neg a$ holds. Assume that the two players will gain most from communication if they resolve whether a or $\neg a$ holds. Assume that player 1 knows the sentence above holds $K_1\varphi$ and that player 2 is not aware of it $\neg A_2\varphi$ (or may even be unaware of $a, \neg a$). Furthermore, player 1 knows *that* $K_1\neg A_2\varphi$. In particular, player 1 may want to send the message φ to player 2, making him aware of it and hopefully prompting player 2 to send back to player 1 that sentence that will determine the value of a – a sentence that exists according to φ. Now, if φ is a potentially useful message that player 1 can send, wouldn't we want to consider player 2 reasoning about the potential *existence* of such a useful message, even when he is unaware of it specifically? If we do, we must have player 2 reason using $\exists^1 x$ and enumerating over sentences with quantifiers.

We note that additional quantifying operators can be added. For example, one can consider an operator $\forall^\omega$ which quantifies over all sentences that include all quantifiers $\forall^n$ $n < \omega$. Obviously, we are still well defined in enumerating over sentences with transfinite induction, but no longer quantify over all sentences in the language. Hence we may want to consider $\forall^{\omega+1}$ and so on with ordinal indexing. This might be useful if communication asks for reasoning about the existence of some quantified sentence of a countable order.

The semantics follow first-order modal logic (cf. [11]) along the lines it was adapted to propositional quantifiers, i.e. quantifying over level-0 sentences. We follow [14] but our valuation domain includes all sentences in the language. We define models $M = \{W, R = \{R_i\}_{i\in I}, \mathcal{A} = \{\mathcal{A}_i\}_{i\in I}, V, \mathcal{V}\}$ where W is a set of states , R_i's are binary relations on W, $\mathcal{A}_i : W \longrightarrow 2^{\mathcal{S}}$ are the set of sentences in our language that i is aware of at state $w \in W$. $V : W \longrightarrow 2^P$ indicates for each state the set of primitive proposition which are true at that state. Finally, $\mathcal{V} : W \times X \longrightarrow S$ assigns at each state to each variable $x \in X$ a sentence in our language $\mathcal{V}(w, x) \in S$.

Truth for formulae is defined inductively as follows:

$(M, w) \vDash p$	if $p \in V(w)$
$(M, w) \vDash \neg\varphi$	if $(M, w) \nvDash \neg\varphi$
$(M, w) \vDash \varphi \wedge \phi$	if $(M, w) \vDash \varphi$ and $(M, w) \vDash \phi$
$(M, w) \vDash A_i\varphi$	if $\varphi \in \mathcal{A}_i(w)$
$(M, w) \vDash K_i^E\varphi$	if $(M, w) \vDash A_i\varphi$ and $(M, u) \vDash \varphi$ for all wR_iu
$(M, w) \vDash \varphi$	if $(M, w) \vDash \varphi[x_1/\mathcal{V}(w, x_1), ..., x_k/\mathcal{V}(w, x_k)]$ where $f(\varphi) = \{x_1, ..., x_k\}$
$(M, w) \vDash \forall^n x\varphi$	if $(M', w) \vDash \varphi$ for all $M' = \{W, R, \mathcal{A}, V, \mathcal{V}'\}$ such that $\mathcal{V}'(w, x)$ is a level n sentence and $\mathcal{V}'(u, y) = \mathcal{V}(u, y)$ for all $u \in W, y \neq x$

Note that the truth value is well defined inductively given (M, w) since we can run a triple induction on sentences, formulae with free variables and n-level quantifiers. Notation-wise we also opted for incorporating $\mathcal{V}$ in M rather than using $\vDash_{\mathcal{V}}$ or $(M, w, \mathcal{V})$. For the purpose of strategic communication we consider the inference rules and axioms comprising of propositional calculus and:

$(K_i^E\varphi \wedge K_i^E(\varphi \implies \phi) \wedge A_i\phi) \implies K_i^E\phi$

$K_i^E\varphi \implies A_i\varphi$

$(\neg\forall^n x A_i x \wedge A_i\neg\forall^n x A_i x) \implies K_i^E(\neg\forall^n x A_i x)$

From φ infer $A_i\varphi \implies K_i^E\varphi$

From φ infer $\forall^n x\varphi$

From φ and $\varphi \implies \phi$ infer ϕ

$\forall^n x\varphi \implies \varphi[x/\phi]$ for every level n formula ϕ

$\forall^n x(\varphi \implies \phi) \implies (\forall^n x\varphi \implies \forall^n x\phi)$

$\varphi \implies \forall^n x\varphi$ if $x \notin f(\varphi)$

$(\forall^n x K_i^E\varphi \wedge A_i\forall^n x\varphi) \implies K_i^E\forall^n x\varphi$

$K_i^E\varphi \implies \varphi$

$(K_i^E\varphi \wedge A_i K_i^E\varphi) \implies K_i^E K_i^E\varphi$

$(\neg K_i^E\varphi \wedge A_i\varphi \wedge A_i\neg K_i^E\varphi) \implies K_i^E\neg K_i^E\varphi$

$\forall^{n+1} x\varphi \implies \forall^n x\varphi$

where these correspond to the system in [14] if we were to add $A_i\varphi \implies K_i^E A_i\varphi$ and consider only the operator $\forall^0$. Since we consider all level n operators we also add the axiom implied by inclusion of range of quantification: $\forall^{n+1}x\varphi \implies \forall^n\varphi$ for every $n = 0, 1, ...$. Finally, we do not impose implicit knowledge of awareness which requires the extra consideration of awareness for deductions of explicit knowledge. See more on this interplay of knowledge and awareness in [6] and [30]. The modifications above allow us to keep the range of A_i to be arbitrary which lets us, for example, consider awareness of only a finite collection of sentences. With this language at hand we turn to our construction of the dynamic communication game.

There are a number of ways we can define a communication game once we set the language. These varieties differ in the extent of reasoning we want to capture in the game form and in the restrictions imposed on the dynamics of reasoning. We discuss the various alternatives below.

3. MESSAGES ACTION SET

The action set available at the message stage at period t of the game is simply her current collection of sentences she can reason about and the manner in which she can send these messages. In particular:

$$S_i^t = \{\varphi | A_i\varphi\}$$

As usual, a pure action is denoted s_i^t, and action profile $s^t \in \prod_i S_i^t$ with mixed action sets $\Sigma_i^t = \Delta(S_i^t), \Sigma = \prod_i \Sigma_i$. A generic action set is denoted S_i. If there is a model M and a state w associated with the game at stage t, we have $S_i^t = \mathcal{A}_i(w)$.

In addition, the player can choose to prefix each message $\varphi \in S_i^t$ as follows:

- **Stating Awareness.** Simply saying "φ". Such a message is denoted φ.
- **Stating Truth/Knowledge.** Stating "φ is true". This message is denoted $\top\varphi$.
- **Questioning Truth/Knowledge.** Asking "Is φ true?". Denoted $?\varphi$.
- **Questioning Variable Value.** For $\varphi = \exists^n x\phi$ asking "What is x such that ϕ ?". Denoted $x?\varphi$.

Hence the set of pure actions for the stage game t is a pair in $\Lambda \times S_i^t$ where $\Lambda = \{\emptyset, \top, ?, x?\}$ is the type of message. We note that stating awareness is always truthful. By definition, a player can only state something she is aware of. We say that stating knowledge of φ is *truthful* if $K_i^E\varphi$. We point out that a player may state something truthful, but she would not view it as such. To see why, note that we allow φ and $K_i^E\varphi$ to hold together with $\neg A_i K_i^E\varphi$. In this case, the player does not reason about her knowledge of φ, hence stating "φ is true" is truthful, although she did not intend it to be. More formally, the intension to be truthful requires the additional condition $A_i K_i^E\varphi$. We also point out that a question requires a notation of whether a future message is an answer. Since answers are in the form of stating knowledge we omit explicit differentiation to whether a statement is an answer as the context should provide it, e.g. sending "φ is true" after being asked "Is φ true?".

4. STATIC GAME FORMS

The choices made in a strategic interaction are based on the evaluation of the outcomes they generate. This introduces two sources of uncertainties, those that are generated by the behavior of other players and those that are intrinsic to the outcomes, i.e., generated by uncertainties about payoffs. The two sources may also interact in the form of uncertainties about others' information or preferences. In strategic communication the payoffs are eventually determined by what the players end up knowing, what they are/become aware of, and the cost of communication. There are various types of payoff functions that might be relevant. Some natural candidates include payoffs that are determined by whether certain events hold and payoffs that are determined by a distance between the players' views of the world. While we discuss the possible payoff function types later on, the game form itself can provide ways to express such uncertainties. We suggest a number of game forms and indicate the uncertainties they allow to formulate. The multiplicity of game forms emerges from the different restrictions we may want to impose on how game theoretic types' beliefs relate to the what the players are aware of and know as expressed in the epistemic model.

Consider, for example, payoffs that are determined by whether a specific sentence is true or not. In this case, assigning probabilities to the truth value of sentences would factor in. In particular, there might be sentences that a player does not eventually know, but is aware of. Hence, estimating the probability these statements hold would be part of estimating the payoffs and essential for implementing a solution. Alternatively, we might be interested in the distance between the set of statements each player is aware of. This would require a player to consider their action as far as how it impacts the awareness of others. Hence, beliefs about this awareness may be called for. Consider the following alternative game forms:

The "base" game form:

D Deterministic. This game form does not incorporate any of the players' beliefs. A player has no "types" or private information issues. The reasoning about other players, or deductions about them are all captured by the strategies and solutions.

The following game forms all deal with a game theoretic representation of uncertainties about events. We begin with three possibilities for defining first order beliefs:

MI1 Model Independent Beliefs – First Order. Each player $i \in I$ has a probability assigned to each member $\varphi \in S_i$. The probabilities are consistent with respect to what the player is aware of and knows, e.g. $K_i^E\varphi$ implies the probability assigned to φ should be 1, $a, a \wedge b \in S_i$ implies the probability of $a \wedge b$ is no higher than the probability of a, $a \wedge b, a \wedge \neg b \in S_i$ implies the sum of their probabilities is no more than 1 even if $a \notin S_i$, and so on.

MC1 Model Consistent Beliefs – First Order. Fix a model M and a state w. The consistency of belief regarding events in S_i does not imply the consistency of beliefs with respect to the model. For example, we might have $a, b \in S_i$ and $a \longleftrightarrow b \notin S_i$. In the model we may have that $a \longleftrightarrow b$ always holds, although neither a nor $\neg a$ always hold. It would be perfectly consistent with respect to what i is aware of and knows to have a and b assigned different probabilities. However, whichever probability measure we use as a prior for the set of states W in the model, it will always induce equal probabilities for a and b, no matter what events we condition on. Hence, requiring that the beliefs over events in S_i be consistent with the model – be generated by some prior over W is a strictly stronger requirement than **MI1**. Formally, given a model M, a probability distribution $\eta \in \Delta(W)$ induces for every formula φ the probability $Pr_\eta^M(\varphi) = \eta(\{w \in W | (M, w) \vDash \varphi\})$. Such a probability distribution is called a *prior*. A *posterior* for player i at w is defined as a probability distribution $\nu(w)$ over the set $\{u \in W | wR_iu\}$. A posterior induces probabilities over formulae in the same manner: $Pr_{\nu(w)}^M(\varphi) = \nu(w)(\{u \in W | (M, u) \vDash \varphi\}) = \nu(w)(\{u \in W | wR_iu \text{ and } (M, u) \vDash \varphi\})$. Condition **MC1** requires that the beliefs player i has over members of S_i correspond to a posterior at the given state of the world. In particular we note that for the given state w we have $S_i = \mathcal{A}_i(w)$.

CP1 Common Prior Beliefs – First Order. The natural strengthening of consistent beliefs generated by a prior is a belief generated by a common prior. This simply states that there exists $\eta \in \Delta(W)$ such that for every w the beliefs of *all* players at w correspond to posteriors of η.

Both **MC1** and **CP1** may raise the issue of whether the additional restriction on a player's belief should not provide the player with additional information. In some sense, if the player's beliefs have to be consistent with some prior, shouldn't his reasoning take this into account? As we mentioned before, the advantage of a game theoretic setting is that we need not explicitly deal with this issue at this point. We can assume that a player inherited beliefs from some prior, or even a common prior, and study the impact of this assumption on the behavior of players. We do keep these distinctions in mind, since if one wishes to provide epistemic characterizations of solutions these conditions require explicit reasoning.

Where there is uncertainty and multiple players one can make a case for high order uncertainties. Here the plot thickens as the players need not be aware about the same events. Our first form allows one player to incorporate beliefs about the beliefs of another player about events the first player may be unaware of.

MII Model Independent Implicit Beliefs – Type Space. Each player has a consistent probability assigned to members of S_i, they have a coherent belief in the sense of [19] and consistent probabilities (in the sense of **MI1**) over the product space of members in S_i with the set of consistent probabilities (in the sense of **MI1**) assigned to members of $\bar{S}_j$ where $\bar{S}_j$ may be any set of sentences a player may be aware of in the language (with no relation to a single model)[1]. Since there may be an uncountable collection of sets $\bar{S}_j$ we would also require a topology on these sets which admits the [19] universal type space construction, i.e. where the space of measures over the collections of possible $\bar{S}_j$'s is compact in the weak* topology, as well as higher order measure spaces.

Note that **MII** requires some explanation of high order beliefs since a player beliefs reside in different state spaces by the variety of $\bar{S}_j$'s considered. For example, player 1 may assign equal probability to player 2 being aware of φ or not, e.g. equal probability to whether $\varphi \in \bar{S}_2$ or not. Assume further that when player 2 is aware of φ he always assigns probability .3 to it. Now, does player 1 believe that player 2 assigns to φ the probability .3 or .15? The answer is that player 1 assigns probability .5 to player 2 assigning probability .3 but that this does not imply that player 1 assigns probability .5 to players 2 assigning a probability to φ which is different from .3. In fact, player 1 assigns probability 0 to player 2 assigning a probability other than .3 to φ. The interpretation is that player 1 assigns probability .5 to player 2 *assigning* a probability to φ and conditional of assigning a probability to φ, that probability is always .3.

MCI Model Consistent Implicit Beliefs – Type Space. Here the type space for high order beliefs is generated by a set of type profiles W and a collection of priors $\eta_i \in \Delta(W)$. This game form has the standard incomplete information game form (without common priors). For example, the probability that player i assigns to the event (φ and $Pr_j(\phi) = .3$) when her type corresponds to w, is $\nu_i(w)(\{u \in W | wR_iu \text{ and } (M, u) \vDash \varphi \text{ and } Pr_{\nu_j(u)}^M(\phi) = .3\})$, i.e. the probability i's posterior assigns at w to states u where φ holds and where the posterior of j assigns probability .3 to ϕ.

CPI Common Prior Implicit Beliefs – Type Space. This is the model consistent implicit beliefs with a common prior $\eta \in \Delta(W)$.

The final collection of game forms we consider introduces high order beliefs that respect the awareness of each player. In this case we have to decide between two ways to address high order awareness. Consider the case where $A_1(A_2\varphi \wedge \phi)$ but $\neg A_1 A_2 \varphi$. The question is whether to use a probabilistic

[1]The only restrictions on $\bar{S}_j$ in this definition are derived from the axioms. For example, we cannot have $K_j^E a \in \bar{S}_j$ while $K_i^E \neg a \in S_i$.

state space where player 1 assigns probabilities to player 2 assigning probabilities to φ or not. Both methods can be used and the definitions follow.

Consider S_i as defined above. Let $S_{i,j} = \{\varphi | A_j\varphi \in S_i\}$. For every two sentences φ, ϕ we write $\varphi \sqsubset \phi$ if φ appears in ϕ, i.e. φ is part of the inductive construction of ϕ in the language. Let $\hat{S}_i = \{\varphi | A_i\phi \text{ and } \varphi \sqsubset \phi \text{ for some } \phi\}$. For the second order sentences we let $\hat{S}_{i,j} = \{\varphi | A_j\phi \in \hat{S}_i \text{ and } \varphi \sqsubset \phi \text{ for some } \phi\}$. Similarly, we define higher order sets $S_{i_1,\dots,i_n}, \hat{S}_{i_1,\dots,i_n}$ respectively. Note that the construction $\hat{S}_i$ essentially allow probabilities to be assigned to every sentence that is used in sentences i is aware of. This corresponds to assigning probabilities to exactly the collection of events that are determined by a set of atomic statements. The corresponding axiom imposed on the language (where awareness of a formulae is derived from awareness of its building blocks as in [6]) would have the two constructions coincide. Note that these structure are similar to those proposed in [7] and [8]. With these structures we can now define beliefs hierarchies as follows.

MIE Model Independent Explicit Beliefs – Type Space. As in the implicit case we need to consider the probabilities that player i assigns to which set of sentences player j is aware of. In the current case, this set of sentences must be a subset of $S_{i,j}$. Hence, player i has a belief over the elements of S_i, over the set of subsets $2^{S_{i,j}}$ and over the possible probabilities j might have over $\tilde{S}_{i,j}$ for every subset $\tilde{S}_{i,j} \subset S_{i,j}$. More precisely, player i's first order beliefs are over S_i, and second order beliefs are over the product of the events from S_i the subsets $S_{i,j}$ for every j and the probabilities over each subset for every player j. Higher order beliefs are defined similarly with coherency required in the sense of [19] as well as the consistency with the properties of the logic system. Once again, the subsets in $S_{i,j}$ must display a topology inducing compact sets of distributions for a universal type space construction.

MIEE Model Independent Explicit Expanded Beliefs – Type Space. Replacing $S_{\dots}$ with $\tilde{S}_{\dots}$ in the definition of **MIE** yields the high order beliefs expanded to all sentences that are part of sentences players are aware of.

MCE Model Consistent Explicit Beliefs – Type Space. A prior η_i as in **MCI** generates probabilities over elements of S_i, but we may well have the sentence $A_j\varphi$ hold at wR_iu (or even w itself) while $\varphi \notin S_{i,j}$. Hence, the beliefs that a posterior ν for a prior η_i generates over which sentences j is aware of, are beyond the support of the second order beliefs. A natural solution is to consider for every u such that wR_iu the set $\mathcal{A}_j(u) \cap S_{i,j}$. Hence the posterior induces a distribution over subsets of $S_{i,j}$ as required. For each sentence in the intersection the second order belief is well defined since player j indeed reasons about the sentences $\mathcal{A}_j(u)$ at u implying that he has a well defined set of marginal probabilities for each $\mathcal{A}_j(u) \cap S_{i,j}$. We have a type spaces that is generated by types over the model, but the high order beliefs on set of statements considered are restricted to subsets of $S_{i,\dots}$.

MCEE Model Consistent Explicit Expanded Beliefs – Type Space. Replaces $S_{\dots}$ with $\hat{S}_{\dots}$ throughout.

CPE Common Prior Explicit Beliefs – Type Space. As defined in **MCE** but with a common prior.

CPEE Common Prior Explicit Expanded Beliefs – Type Space. As defined in **MCEE** but with a common prior.

Yet another alternative to all the game forms above would be to consider beliefs over elements not generated by the sets S_i, but rather by considering the set of sentences that player i explicitly knows she is aware of, i.e., $\{\varphi | K_i^E A_i\varphi\}$. This essentially fixes the set of sentences for every uR_iw, although one should not interpret this as *knowing* the probability since the probability considers a set that may be strictly smaller than the set considered by the model at every state.

As we mentioned earlier, some forms may coincide with additional axioms imposed on the language. In particular, awareness of a sentence implying awareness of its components will combine the extended and standard explicit beliefs. Similarly, knowledge of awareness will avoid the varying domains of subsets of S_i that a model type space needs to consider for players' posteriors.

To conclude the variety of game forms that might be considered we point out the tension between the epistemic model and the game theoretic constructs. For example, in **MCI** the game form has the player assign probabilities to events that the epistemic model assumes is beyond the reasoning ability of the player. But, this tension exists even in the game form aiming to be most aligned with the epistemic model, such as **MCE**, or even the game form that ignores high order beliefs such as **MC1**. In all of these we associate beliefs with a player, without a formal logical treatment of reasoning about these beliefs. The beliefs are outside our language. We point out that this is not necessarily a deficiency of the game theoretic approach. In fact, borrowing from choice theory, probabilities may serve as implicit mathematical formulation for behavior. We do not assume that our subjects reason about their subjective probabilities yet we can describe measurements of behavior that demonstrate they act *as if* they maximize in a structure involving probabilities. In our case as well, we might ask for a game theoretic solution where players behavior is *as if* they respond to probabilistic beliefs. Nevertheless, we do recognize that solution based on probabilities assigned to events beyond the scope of reasoning require not only a stretch of imagination, but may not allow an empirical test via revealed behavior.

5. LOGIC DYNAMICS

A message sent from one player to another can impact both the epistemic description of their reasoning, the game theoretic state as well as the beliefs or behavior associated with a solution concept to the game. In this subsection we consider the dynamics of the game after a message is sent, i.e., the reactions to a message received. These reactions may logic dynamics restriction, e.g. we may want to impose that all players become aware of the message sent. They also may be strategic reactions, e.g. a player can decide whether the message about the truth of a sentence is indeed true. This decision is interpreted as a choice of action, the action being *which* set of sentences should the player be aware of and know in the next stage of the game. We implement this approach by introducing the logical dynamics conditions as constraints over the strategies the players may hold. The actual dynamics of the model and state, or the set of sentences the players reason about, is the result of the players choosing what they become aware of and know within the logical dynamics constraints. Treating the dynamics of knowledge and awareness as a strategic choice makes it easy to analyze beliefs about these dynamics via solution concepts.

While the dynamics of players awareness and knowledge is game theoretic in our setting, recent work in logics by [13], [15], [16], [28], [29] and [30] provides a collection of models for capturing the dynamics of awareness within the language. There are multiple principles and properties that can be imported into the game theoretic setting. These also interact with the established and very rich literature on the dynamics of knowledge and reasoning. See, notably, [1], [21], [24], [27], [31] and the references there. In addition one can find a vast literature dealing with the logics surrounding questions we will focus on the dynamics of stating awareness and knowledge with only cursory attention to the dynamics induced by questions.

The dynamics of logical restrictions are defined as a mapping of the history of models, states and messages to the eligible set of models and states, i.e., it capture the possible constraints over how the static model may appear at each period t as a function of what was thought and said up to period t. The dynamics are not aimed at describing the strategic dynamics in the game, instead they confine the game forms we consider by stating which models and states are allowed at every period. A *dynamics* D is a collection of functions $D^1, D^2, ...$ such that

$$D^t((M^1, w^1), (\lambda^1, s^1_{i_1}), ..., (M^t, w^t), (\lambda^t, s^t_{i_t})) \subset \mathcal{M}$$

where $\mathcal{M} = \{(M, w)|M$ is a model in the language and $w \in W$ for $M\}$. We note that for every t we assume $s^t_{i_t}$ is in the action set for i_t given (M^t, w^t). We also assume that the argument (M^t, w^t) in D^t has a domain restricted to the range of D^{t-1} and that all conditions are with respect to this domain.

Next we turn to a variety of potential restrictions on the dynamics of reasoning. We say that the two pairs (M, w) and $(\bar{M}, \bar{w})$ are *equivalent* and denote $(M, w) \sim (\bar{M}, \bar{w})$ when $(M, w) \vDash \varphi$ if and only if $(\bar{M}, \bar{w}) \vDash \varphi$ for all φ. We say that they are *reasoning equivalent for i* and denote $(M, w) \sim^R_i (\bar{M}, \bar{w})$ when $(M, w) \vDash \varphi$ if and only if $(\bar{M}, \bar{w}) \vDash \varphi$ for all φ of the form $A_i\phi$ or $K^E_i\phi$. We say they are *reasoning equivalent* denoted $\sim^R$ if they satisfy $\sim^R_i$ for all $i \in I$. If we only require that $(M, w) \vDash \varphi$ if and only if $(\bar{M}, \bar{w}) \vDash \varphi$ for all φ of the form $A_i\phi$ for all players, we say that they are *awareness equivalent* and denote $(M, w) \sim^A (\bar{M}, \bar{w})$

DMI Model Invariance. D is model independent if for every t and for every $(M, w) \sim (\bar{M}, \bar{w})$ we have that D^t is invariant to replacing (M, w) with $(\bar{M}, \bar{w})$.

DRI Reasoning Invariance. D is invariant to reasoning equivalent model-state pair.

DAI Awareness Invariance. D is invariant to awareness equivalence.

We note that **DAI** implies that D is determined by the set of messages available to the players at each stage of the game and would allow for a definition of dynamics based on action sets and not the underlying models and states. However, **DRI** and **DAI** are distinct since the knowledge of a player may not be fully reflected in their action set as discussed above. Had we added the axiom $K^E_i\varphi \Longrightarrow A_iK^E_i\varphi$ the two properties would have coincided.

DM Markovian. D is Markovian if D^t depends only on its last two arguments $(M^t, w^t), (\lambda^t, s^t_{i_t})$.

The Markovian dynamics assumes that the current stage model and state together with the current message encapsule all the restrictions on how the next stage static model and state may look like. It does not imply that the strategies in the game need to be Markovian, it all considers how the restrictions on the logic evolve.

The next property corresponds to the case where the only change allowed over time is to the reasoning of the players. Hence, the truth value of the atomic statements does not change over time. We say that (M, w) and $(\bar{M}, \bar{w})$ are *propositionally equivalent* and denote $(M, w) \sim^P (\bar{M}, \bar{w})$ when $(M, w) \vDash p$ if and only if $(\bar{M}, \bar{w}) \vDash p$ for all atomic formulae p.

DPNS Dynamic Preservation of Non-Epistemic State. For all t, for every $(M, w) \in D^t(..., (M^t, w^t), (\lambda^t, s^t_{i_t}))$ we have $(M, w) \sim^P (M^t, w^t)$.

The following two properties concern a process that is more deterministic. These conditions correspond to cases where one is able to precisely define how the players react to statements. It would still allow for strategic choice, but once statements are made, the evolution of reasoning is determined.

DDM Deterministic Model. For all t, every $D^t(..., (M^t, w^t), (\lambda^t, s^t_{i_t}))$ is an equivalence class of the relation $\sim$ on $\mathcal{M}$.

DDR Deterministic Reasoning. As above with the range being an equivalence class of the relation $\sim^R$ on $\mathcal{M}$.

DDA Deterministic Awareness. As above for $\sim^A$.

note that it would seem reasonable to confine a deterministic dynamics to a dynamics of the relevant equivalence class, i.e. assume that D satisfies **DMI**, **DRI**, or **DAI** whenever it satisfies **DDM**, **DDR**, or **DDA** respectively.

For the next collection of properties, it is convenient to introduce the notation $D^t(..., (M^t, w^t), (\lambda^t, s^t_{i_t})) \vDash \varphi$ indicating $(M, w) \vDash \varphi$ for all $(M, w) \in D^t(..., (M^t, w^t), (\lambda^t, s^t_{i_t}))$.

DMR Monotonic Reasoning. For all t, i, and φ such that $(M^t, w^t) \vDash A_i\varphi$ (resp. $(M^t, w^t) \vDash K^E_i\varphi$) we have $D^t(..., (M^t, w^t), (\lambda^t, s^t_{i_t})) \vDash A_i\varphi$ (resp. $\vDash K^E_i\varphi$).

DMA Messages Awareness. For all $t, i_t = i, j$ we have $D^t(..., (M^t, w^t), (\lambda^t, s^t_{i_t})) \vDash A_js^t_{i_t}$.

DKMA Knowledge of Message Awareness. As above but with $D^t(..., (M^t, w^t), (\lambda^t, s^t_{i_t})) \vDash K^E_{j_1}...K^E_{j_k}A_js^t_{i_t}$ for all k and $j_1, ..., j_k \in I$.

6. COMMUNICATION GAMES

The game dynamics depends on the specific static game form chosen for the game. We provide the description of the game for each of the three cases: **D**, **MII**, and **MCI**. The definition for other static forms can be derived accordingly. We also point out that the conditions on the logic dynamics imposed by D create a restriction on the choices a player has in updating her awareness and knowledge.

Let $R_i \subset \{\varphi \in \mathcal{S} | \varphi = A_i\phi \text{ or } \varphi = K_i^E\varphi\}$ denote the sentences describing player i's reasoning. We assume throughout that R_i is logically consistent, and denote by $R_i^M(w) = \mathcal{A}_i(w) \cup \{K_i^E\varphi | (M, w) \vDash K_i^E\varphi\}$ the set R_i generated at w in a given model M. Let $\mathcal{R}$ denote the set of all possible R_i's in the language.

GD Deterministic Communication Game. The game begins with a collection $R = \{R_i\}_{i\in I}$. There is no a priori assumption that all the R_i's are consistent, i.e. that there is a common model and state (M, w) generating them. The pure behavior strategies of each player in the dynamic game are defined as follows. If it is i's turn to send a message at period t ($i_t = i$) we have a function $m_i^t(R^1, (\lambda^1, s_{i_1}^1), ..., R^t) \in \Lambda \times S_i^t$. After a message has been sent, all players revise their reasoning using strategies of the form $r_i^t(R^1, (\lambda^1, s_{i_1}^1), ..., R^t, (\lambda^t, s_{i_t}^t)) \in \mathcal{R}$

We impose the following conditions on r_i^t:

ARI Action Reasoning Invariance. For all t, i the choices r_i^t and m_i^t depend only on $(R_i^1, (\lambda^1, s_{i_1}^1), ..., R_i^t, (\lambda^t, s_{i_t}^t))$ (with the latter term removed for m_i^t), i.e. each player revises their reasoning and sends messages based only on their own past reasoning and the messages they observed.

AC Action Consistency. For every i, t and realization $(R^1, (\lambda^1, s_{i_1}^1), ..., R^t, (\lambda^t, s_{i_t}^t))$ there exist a pair (M, w) and a sequence $(M^1, w^1), ..., (M^t, w^t)$ such that $(M, w) \in D^t((M^1, w^1), (\lambda^1, s_{i_1}^1), ..., (M^t, w^t), (\lambda^t, s_{i_t}^t))$, $R_i^k = R_i^{M_k}(w_k)$ and $r_i^t(R^1, (\lambda^1, s_{i_1}^1), ..., R^t, (\lambda^t, s_{i_t}^t)) = R_i^M(w)$. In words, the players choice of reasoning revision respects the restrictions provided by the logic dynamics D.

We note that while **AC** guarantees that the set of sentences each player chooses is logically consistent, it does not imply that there exists a pair (M, w) that can be associated with the simultaneous choice of *all* players. For example, a player may choose to know something that another player may know is false. Alternatively, a player may believe a lie the sender knows to be a lie. In this case, the game form defined by **GD**, as well as the following game form **GMII** which is independent of models, both suit the bill, however the game is allowed to evolve into a setting where players can only be associated with disjoint sets of models.

There are a number of potential conditions that can guarantee the players always have a shared model and a state that corresponds to all the sentences they choose. The simplest of those would be to assume that the players have all begun at a single model and that no one updates their explicit knowledge at all. A more realistic setting would have the players only update their knowledge to logical deductions. Yet another addition would allow for **DKMA** under the assumption of **DMR** (making sure that knowing the awareness of others implies that they are aware). A more specialized case would allow players to add the messages $\top\varphi$ in addition to logical deductions, but restrict all players to tell the truth – maybe not the most practical setting. Finally, one can use a variant of the language by adding implicit knowledge operators. These act as standard K_i knowledge operators for each player and interact with explicit knowledge and awareness by $K_i^E\varphi = K_i\varphi \wedge A_i\varphi$. See [6] for a discussion of implicit knowledge and [14] for its integration with quantifiers. With K_i we can restrict the player from explicitly knowing any statement other than statements she implicitly knows. Obviously, this would still imply a player could not know a lie which in turn introduces some unclarity as to how a player perceives $\top\varphi$ when she is not allowed to consider it as true.

Our next game corresponds to the static form **MII**. Consider an initial R as above and a type space corresponding to **MII**, i.e. a type space $\mathcal{T} = \prod_{i\in I} \mathcal{T}_i$ such that $\tau_i \in \mathcal{T}_i$ induces a probability over members of S_i, beliefs over player j's beliefs over possible subsets of sentences, as well as correlations between these probabilities, such that all beliefs are logically consistent and the high order beliefs are coherent.

GMII Model Independent Implicit Beliefs Game. The game begins with a collection $R^1 = \{R_i^1\}_{i\in I}$ and type space $\mathcal{T}^1$. If it is i's turn to send a message at period t we have a function $m_{\tau_i^t}^t(R^1, (\lambda^1, s_{i_1}^1), ..., R^t) \in \Lambda \times S_i^t$. After a message players revise their reasoning with $r_{\tau_i^t}^t(R^1, (\lambda^1, s_{i_1}^1), ..., R^t, (\lambda^t, s_{i_t}^t)) \in \mathcal{R}$. Note that we retain only the type at period t. An alternative would retain all history of types and consider the revision of reasoning conditional on the history of types as well. Throughout we assume that τ_i is consistent with R_i. We assume both **ARI** and **AC** hold.

We point out that the game **GMII** allows players to revise into contradictory knowledge, or believe lies, but it does not allow players to assign probabilities to these possibilities. For example, two players may revise into $K_1^E\varphi \in R_1^t$ $K_2^E\neg\varphi \in R_2^t$. Their types at period t, however, will assign probability zero to $\neg\varphi$ and φ respectively, as well as to $K_2^E\neg\varphi$ and $K_1^E\varphi$ respectively. A type space $\mathcal{T}^2$ could still accommodate this update since the consistency and coherency of each player's beliefs can hold by a player assigning zero probability to the actual type of the other player.

The last game we consider is generated by a model in $\mathcal{M}$ and posterior beliefs which generate the high order type space in the static setting. The difference from the model independent case is that models used for high order reasoning can now be related to the dynamic logic restrictions D. In this game we also allow the game theoretic type-spaces to diverge from the first period (e.g. we allow players' type spaces to diverge from the fist period).

GMCI Model Consistent Implicit Beliefs Game. The game begins with a collection $R^1 = \{R_i^1\}_{i\in I}$ and a triplet $\{(M^1, w^1, \nu^1)_i\}_{i\in I}$ for each player. Such that $R_i^1 = R_i^{(M^1)_i}((w^1)_i)$. As in **GMII** strategies are defined over last period type: $m_{(w^t)_i}^t(R^1, (\lambda^1, s_{i_1}^1), ..., R^t) \in \Lambda \times S_i^t$ and $r_{(w^t)_i}^t(R^1, (\lambda^1, s_{i_1}^1), ..., R^t, (\lambda^t, s_{i_t}^t)) \in \mathcal{R}$ where $(w^t)_i$ stands for player i's type at w in the model $(M^t, w^t)_i$ under the priors $(\nu_j)_i$ associated with each player j. We require that $R_i^t = R_i^{(M^t, w^t)_i}$ for all i, t. We assume both **ARI** and **AC** hold and note that one can assume **AC** holds with the specific models $(M^t, w^t)_i$, i.e. that the model used in each period as the type space for a player satisfies the condition imposed by D.

It is important to point out that while the objects $(M^t, w^t, \nu^t)_i$ may be constrained by the game form above and D, they are not uniquely determined. We assume that for every specific game of the form **GMCI** the selection of the distinct $(M^t, w^t, \nu^t)_i$ for every player for every period is unique and completely determined, much like an incomplete information game with a given state of the world.

A similar extension of the static forms which are defined by a model **CPI**, **MIEE**, **MCE**, **MCEE**, **CPE**, **CPEE** follows the definition of **GMCI**. For all these game forms we may want to require that the players game theoretic type is always generated by a single model at a single state. This, naturally, guarantees that the reasoning of players R_i^t is always consistent. We call such a restriction an *objective* game. The variant for **GMCI** will be:

GOMCI Objective Model Consistent Implicit Beliefs Game. The game begins with a collection $R^1 = \{R_i^1\}_{i\in I}$ and a single $\{(M^1, w^1, \nu^1)$. Such that $R_i^1 = R_i^{M^1}(w^1)$. Messages: $m^t_{w^t}(R^1, (\lambda^1, s^1_{i_1}), ..., R^t) \in \Lambda \times S_i^t$. Reasoning: $R_i^{t+1} = r^t_{w^t}(R^1, (\lambda^1, s^1_{i_1}), ..., R^t, (\lambda^t, s^t_{i_t}))$ where w^t stands for player i's type at w in the model (M^t, w^t) with priors ν_j associated with each player j. Here, there is a single model-state pair at every period shared by all players and consistency is guaranteed by $R_i^t = R_i^{M^t}(w^t)$ for all t, i.

In addition to the pure behavior strategies defined for the game form above we should consider mixed strategies. These are assumed to be standard, in the sense that a message may drawn at random, or that a player randomly picks one set R_i or the other. The realizations are determined by the appropriately randomized mixed profiles. We note, that the randomization over the reasoning choice is sure to generate multiple distinct models even if an objective model consistent setting is considered, each model corresponding to the different realizations of the reasoning dynamics. Nevertheless, the extensions of our conditions hold, where one may ask for each realization path to conform to the definition in **GOMCI**, i.e., to a joint model generating players' types. Similarly, we expect each realization to satisfy both **ARI** and **AC** in all game forms.

We conclude the game forms definitions with a description of the payoffs functions for communication games. For simplicity we will consider finite T-period undiscounted game. The payoffs are determined by two aspects of the game, the messages sent and the final state of reasoning of all players. The cost of sending messages is assumed to be separable in time and defined by $c : \Lambda \times \mathcal{S} \longrightarrow \mathbb{R}_+$. We assume that the cost of sending a message is born by all players. Furthermore, if the sets S_i^t are infinite, it makes sense to define a cost that guarantees the existence of best responses, as in the case when players reason about all logical deductions. We may want to have $c(\lambda, \varphi)$ increase with the length of φ, e.g. by setting $c(\lambda, \varphi) =$ the number of symbols that appear in φ. The final state of reasoning is defined as the realization of r_i^T which we can denote by R_i^{T+1}. Hence, the payoff from reasoning for each player is a function $p_i : \prod_{j\in I} \mathcal{R} \longrightarrow \mathbb{R}$. We point out that an alternative payoff function definition would incorporate the state of some terminal model in the game. In particular, we might be interested in payoffs that condition on whether a sentence holds true or not. This is relevant for the games that interact with models and would augment the payoff function to depend on M, w as well, or on whether $M, w \vDash \varphi$ for some M, w associated with the termination of the game. We note that such an extension corresponds to an incomplete information game regarding the truth value of proposition even before we consider the dynamic reasoning game. We point out that restricting the payoff as we did guarantees that the payoff is a function only of the sentences the players actually end up reasoning about.

7. SOLUTIONS

The basis for most solution concepts is the notion of best response. This simple notion already introduces multiple interpretations since the players can be considered to be playing best responses to the actual strategies other players are playing, or to their perception of the strategies other are playing. Moreover, a given player may have a behavioral best response at each period which does not match a best response strategy for the whole game when the player may not be able to reason correctly about the impact of choices due to unawareness. However, our definitions of the various communication games are all close to standard games. In the game **GD** we have a standard game where the players get to choose dynamically the set of messages they can send, under exogenous restrictions over the chosen sets. Hence, it is a dynamic game with players who stochastically choose the element of the sets R_i under a set of exogenous conditions and such that S_i becomes the messages choice set. In the other game forms (both model dependent and model independent), we have a sequence of incomplete information games, where the selection of sets R_i determines the type space in the next stage of the game. Once again, this occurs under a variety of potential conditions that exogenously restrict the choices, type spaces, or relates the type spaces generated by different players. It is here that the solution is required to address a diverging incomplete information game structure. We define the best response correspondences for the communication game **GMCI** these are easily extended to dynamic types, or models, for the other games.

Consider a strategy for each player, i.e. functions $\rho^t_{(w^t)_i}$ for $t = 1, ..., T$ with a type $(w^t)_i$ corresponding to $(M^t, w^t, \nu^t)_i$ and a function $\mu^t_{(w^t)_{i_t}}(R^1, (\lambda^1, s^1_{i_1}), ..., R^t)$ for the appropriate message sender at t, which correspond to a mixture of pure strategies $r^t_{(w^t)_i}$ and $m^t_{(w^t)_{i_t}}$, respectively. Let ρ_i, μ_i denote player i's strategy, and ρ, μ all players strategy profile. Player j is said to be playing an *implicit best response* in ρ, μ if player j's expected payoff generated by the players playing ρ, μ is at least as high as the expected payoff generated by $\rho|\bar{\rho}_j, \mu|\bar{\mu}_j$, i.e. player j has no alternative strategy that will increase his payoff keeping the strategies of the other players fixed. Note that the payoffs are well defined for a given ρ, μ as the description of the game specifies the appropriate $(M^t, w^t, \nu^t)_i$ for every contingency. Naturally, an implicit best response has the players play what must seem to them sub-optimal in many cases. While they play a best response to the strategies others use, those strategies respond to a different state space whenever $(M^t, w^t)_i \neq (M^t, w^t)_j$. We could alternatively ask that players play an *explicit best response*, this would be defined as each player playing a best response to strategies equivalent to ρ_j, μ_j played in a game where each player j is assigned the type spaces $(M^t, \nu^t)_i$ and the state w^t is picked according to $(\nu_i^t)_i$. Such a definition would require each $(M^t)_i$ to be rich enough initially so that the mappings ρ_j, μ_j could be extended, and this simultaneously for all players. Furthermore, it should allow this for the $(M^t)_i$ that are assigned for every alternative strategy $\bar{\rho}_j, \bar{\mu}_j$ of player i since we need to consider a best response. Even so, there may not exist an explicit equilibrium since the players are essentially playing best responses while considering different incomplete information games. Finally, we point out that the game **GOMCI** does not suffer from these issues and has a well defined explicit best response. While an equilibrium can be defined in this case, we point out that this equilibrium still may be suffering from the usual problems of a lack of common priors. Moreover, we still must adopt an "as if" interpretation of behavior since the strategies are driven by models that include events the players cannot reason about.

8. CONCLUSIONS

We provided a construction of communication games that is based on incomplete information games or a standard normal form game in the case of **GD**. In these games, player choose messages and revise their knowledge and awareness. The revision is assumed to be a strategic choice. Conditions on the dynamic logic are introduced as restrictions of the strategic revision. A new incomplete information game (standard game for **GD**) begins in each period of the communication game. The players choose messages based on their reasoning, choose their revision, and so on. This construction transferred the reasoning about the interpretation of messages, as well as high order reasoning about how others revise their reasoning, to the game theoretic game form. This allows for the strategic choice of message to incorporate the strategic reasoning revision in a game setting. Moreover, probabilities are easily tacked on to the game side as players require them for evaluating expected payoffs.

We point out that our approach is based on providing a clear boundary between game theoretic modeling and logics in a setting where the two are intermingled. It should be noted that logic dynamics has made great strides to incorporate game theoretic features, cf. [1], [21], [24], [27] and [31] for some major advances. Nevertheless, we hope that leveraging the advantages both disciplines (as was successfully done in logic games, cf. [25] and the references there), will complement the explicit logical modeling of strategic interaction on the one hand and the use of game theoretic formal machinery in logics on the other.

9. ACKNOWLEDGMENTS

The author thanks David Kreps for insightful conversations and suggestions.

10. REFERENCES

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