

Value-based Contraction: A Representation Result

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ABSTRACT

Sven-Ove Hansson and Erik Olsson studied in [6] the logical properties of an operation of contraction first proposed by Isaac Levi in [11]. They provided a completeness result for the simplest version of contraction that they call Levi-contraction but left open the problem of characterizing axiomatically the more complex operation of value-based contraction or *saturatable contraction*. In this paper we propose an axiomatization for this operation and prove a completeness result for it. We argue that the resulting operation is better behaved than various rival operations of contraction defined in recent years.

Categories and Subject Descriptors

I.2.4 [Artificial Intelligence]: Knowledge Representation Formalisms and Methods; F.4.1 [Mathematical Logic]: Modal Logic

General Terms

Theory

Keywords

Belief revision, AGM, contraction, epistemic value, decision theory

1. INTRODUCTION

Willard Van Orman Quine pointed out in *Two Dogmas of Empiricism* that when we engage in the reevaluation of a system of beliefs as a result of a recalcitrant experience we have ‘[...] a natural tendency to disturb the total system as little as possible’. The idea that revisions, when deliberate, tend to deploy this principle of minimal mutilation is probably quite old and pre-dates Quine’s remark.

The contemporary research in belief change is also based on implementing this principle of minimal change. The principle has various names and it has been invoked frequently as a motivation for the development of precise accounts of belief change. It is important to take into account, nevertheless, that every exercise in minimization presupposes that one is minimizing some index over

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a feasible set. It has been controversial though what is the relevant index in contraction and what should be the feasible set over which one performs the minimization.

Peter Gärdenfors proposes in [4] that the relevant index in question is information, where information is measured in terms of propositional content. So, the idea is that when one gives up a belief one should strive to minimize the information loss incurred in the corresponding operation of *contraction*.

One natural idea that was proposed early on by the AGM trio (Carlos Alchourrón, Peter Gärdenfors and David Makinson) is to look at the so-called *remainder sets* of a set of sentences representing current beliefs (a *belief set*). What is a remainder set? Let K be a closed theory representing your commitments to full beliefs. Let α be a sentence in K that one wants to contract. Then one can look at all the maximal subsets of K that fail to entail α . These *maxi-choice* contractions will successfully minimize information loss in contraction. But maxi-choice contractions are not very well behaved when dealing with closed theories and except from applications in formal learning theory ([17]) almost nobody appeal to them.¹

One skeptical move is to adopt the intersection of the remainder sets of K and α (usually denoted $K \perp \alpha$). This is also a draconian strategy that almost everybody rejects. In fact when one contracts a sentence α from a theory K this type of contraction yields the intersection of K with the logical consequences of $\neg\alpha$ in all but a limit case. So, if one expands this contracted state by $\neg\alpha$ one gets exactly the logical consequences of $\neg\alpha$. If we understand the revision of K with $\neg\alpha$ as the process of first contracting α from K and then adding $\neg\alpha$ set theoretically to the resulting contraction with α , we should conclude that the result of revising K with the input $\neg\alpha$ yields the logical consequences of the input $\neg\alpha$ ignoring completely the background theory K , which seems unreasonable. So, the AGM trio stabilized in an Aristotelian middle: they proposed to select a subset from $K \perp \alpha$ and take the intersection of the elements of this set. This is the approach to contraction usually called *partial meet contraction*.

Moreover, AGM proposed to impose constraints on the selection function used in partial meet contraction. Here we have the idea of minimization (maximization), The idea is to select optimal elements from $K \perp \alpha$, i.e. the elements of $K \perp \alpha$ that dominate all elements of the set. Obviously the feasible sets used in this exercise of maximization (minimization) are remainder sets of the form $K \perp \alpha$, for some α .

The set of postulates that is complete with respect to this construction procedure even when no relation is used to rationalize the

¹Maxi-choice contraction plays also an important role when contracting unclosed set of sentences, usually called *belief bases* in the literature. See the work done in this area by Sven Ove Hansson in [7]. In this paper we are dealing only with contraction of theories.

selection function includes some controversial postulates like *recovery*. Suppose that you contract your current belief set K with a sentence α that belongs to it. After that you add set theoretically α to this contracted set and close under consequence. Recovery legislates that when one performs this operation one returns exactly to K . Candidate counterexamples to recovery abound in the literature.²

Enter in the story the philosopher Isaac Levi. He has repeatedly complained that the whole idea of partial meet contraction is fundamentally misguided. Levi preserves the idea that some relevant epistemic index is minimized in contraction over some feasible set (the notion of feasibility used here is the standard one used in decision theory). But he claims as well that the index used by AGM is wrong as well as the feasible set used to optimize. On the one hand one does not merely minimize information loss but what he calls *informational value*. This is a more complex index summarizing different aspects of value that might be relevant in contraction (simplicity or elegance are examples of relevant aspects to take into account). As a result one can have that two sets might carry equal informational value even when one set carries more content than the other. So, a revision (contraction) might look rather ‘severe’ under the point of view of mere information loss but be optimal under the point of view of minimizing the loss of informational value.

On the other hand Levi claims that the feasible set one should take into account to optimize should be less restrictive than the ones recommended by AGM. He proposes saturatable sets that might be strict supersets of the corresponding remainder sets.³ Our goal in this paper is to study the logical properties of the corresponding operation of saturatable contraction. In particular we intend to solve a problem left open by Sven Ove Hansson and Erik Olsson in [6]. We intend to propose an axiom system that fully characterizes the most sophisticated contraction operation defined formally in [6]. This operation of value-based contractions is in turn inspired by (but might not reflect faithfully) the ideas that Levi presented in [11]. If there are indeed differences between the notions presented informally in [11] and the formalized notions introduced in [6] and [7], it would be interesting to study this issue in detail.⁴ But this is not our goal in this paper. Our results extend the work presented in [6] and [7] and therefore we presuppose the definitions of value-based contraction presented in these writings. In our view these

²In spite of the fact that candidate counterexamples to recovery abound some of them remain controversial. For example, David Makinson has argued against the validity of some of these examples in [16]. This paper examines the principal alleged counterexamples to the recovery postulate for operations of contraction on closed theories, and shows that the theories considered are implicitly ‘clothed’ with additional justificational structure. Recovery remains appropriate for ‘naked’ closed theories. A rebuttal is presented by Isaac Levi in [12]. We do not want to elucidate this issue here. We will assume that the role of recovery is at least controversial and we will explore alternative options from a logical point of view.

³As we make explicit in various places below we will adopt definitions of saturatable sets commonly used in the logical literature (especially in [6]). It is unclear whether these definitions capture all the relevant aspects of contraction suggested by Levi in various writings. We think that these definitions are nevertheless interesting and deserve to be studied logically. It is important to emphasize that in this article we do not attempt to provide a complete formalization of the many ideas on contraction expressed by Levi in a multiplicity of papers and books during the last 15 years or so. We have the more modest goal of solving a logical problem left open in [6].

⁴Levi suggested in a personal communication that perhaps there is such gap but the detailed consideration of this issue is left for future work.

definitions are formally adequate and philosophically interesting. So, we think that the logical scrutiny of these notions is worthwhile independently of the possibility of formulating alternative versions of central notions like saturatable set and value-based contraction.

The techniques used in the proof will be, in spite of the conceptual differences with partial meet contraction, reminiscent of the techniques used by AGM to prove their result. Even when transposed into a different domain, the proof techniques used by AGM hold robustly in the new domain.

After presenting the main formal result we compare the resulting notion of contraction with some of the most salient alternatives and conclude that the new notion of contraction is characterized by axioms that for the most part are less controversial than some of the axioms used in these alternatives (mainly the notion of contraction commonly known as *severe withdrawal* and the classical AGM operator).

In a recent influential survey on the logic of belief revision ([8]) Sven Ove Hansson argues that:

The construction of a plausible operation of contraction for belief sets that does not satisfy Recovery is still an open issue.

The argument is based on criticizing some of the properties of various theories that do not satisfy Recovery. Our formal work in this paper shows that the notion of value-based contraction we are proposing lacks many of the shortcomings of these alternative theories. Therefore we propose it as the best available candidate to solve the open problem indicated in Hansson’s article.

2. AGM CONTRACTION

The AGM trio model belief states as belief sets, which are defined as sets of sentences closed under logical consequence. The AGM model is primarily concerned with three types of belief change, namely, *expansion*, *contraction*, and *revision*. Expansion is modeled as set theoretical addition followed by closure under consequence and revision is reducible to contraction via the so-called Levi identity according to which the revision of a theory K with the sentence α is constructible as the contraction of K with $\neg\alpha$ expanded with the singleton α . Thus here we will merely focus on the operator of contraction proposed by AGM. The material included in this section presents in a compact way the main results in [1]. Readers interested in details about these proofs should consult this classical paper.

In this paper, we will assume a classical propositional language L . The beliefs of a rational agent are represented by sentences in L . As usual, we also assume that the language L is closed under the classical truth-functional connectives. We identify the underlying logic with the Tarskian consequence operator $Cn : 2^L \rightarrow 2^L$ which is defined as a function such that for any subsets X and Y of L :

- $X \subseteq Cn(X)$. (**Inclusion**)
- If $X \subseteq Y$, then $Cn(X) \subseteq Cn(Y)$. (**Monotony**)
- $Cn(X) = Cn(Cn(X))$. (**Iteration**)

Moreover, we assume that the consequence operator Cn obeys the following three properties:

- If α can be derived from X by classical logic, then $\alpha \in Cn(X)$. (**Superclassicality**)
- $\beta \in Cn(X \cup \{\alpha\})$ if and only if $\alpha \rightarrow \beta \in Cn(X)$. (**Deduction**)

- If $\alpha \in Cn(X)$, then $\alpha \in Cn(X')$ for some finite subset X' of X . (**Compactness**)

Equipped with this operator Cn , a belief set K can be formally defined as $K = Cn(K)$. In logical parlance belief sets are known as *theories*. The usual epistemological interpretation of theories is as *commitment sets* representing the epistemic commitments of a rational agent ([11]).

The first step used to construct contractions of a belief set K is to focus on the maximal subsets of K that do not imply α , which guarantees minimal loss of information in the subset sense.

DEFINITION 1. Let $K \subseteq L$ be a theory. Then the α -remainder set of K , denoted by $K \perp \alpha$, is the set such that $X \in K \perp \alpha$ if and only if: (i) $X \subseteq K$; (ii) $\alpha \notin Cn(X)$; (iii) There is no set X' such that $X \subset X' \subseteq K$ and $\alpha \notin Cn(X')$.

From this definition, we can directly derive the following two properties of remainder sets: (i) $K \perp \alpha = \{K\}$ if and only if $\alpha \notin K$, and (ii) $K \perp \alpha = \emptyset$ if $\models \alpha$. Compactness further guarantees that $K \perp \alpha = \emptyset$ only if $\models \alpha$. It is well known that the remainder sets of belief sets behave quite well from several perspectives, and have many nice and useful properties. Here we introduce several important properties of remainder sets that will be referred to repeatedly in the foregoing sections.

PROPOSITION 2. Let K be a theory. The following properties hold:

- (i) If $X \subseteq K$ and $\alpha \notin Cn(X)$, then there is some X' such that $X \subseteq X' \in K \perp \alpha$. (Upper bound property)
- (ii) For any sentences $\alpha, \beta \in K$, $K \perp (\alpha \wedge \beta) = K \perp \alpha \cup K \perp \beta$.

We have enough elements now to introduce the main operation of contraction proposed by AGM: *partial meet contraction*. The central idea of this operation is to make a selection among the set of all maximal non α -implying subsets of K ; and then take the intersection of such a selection. A selection function is introduced in order to make this selection. In general, the concept of selection function used by AGM can be formally defined as follows:

DEFINITION 3. Let K be a theory. A selection function for K is a function γ such that for all sentences α : (i) If $K \perp \alpha$ is non-empty, then $\emptyset \subset \gamma(K \perp \alpha) \subseteq K \perp \alpha$; (ii) If $K \perp \alpha$ is empty, then $\gamma(K \perp \alpha) = \{K\}$.

Then partial meet contraction can be defined as follows:

DEFINITION 4. Let K be a theory. An operation $-$ is a partial meet contraction if and only if there is a selection function γ for K such that for all sentences α : $K - \alpha = \bigcap \gamma(K \perp \alpha)$.

It follows from these three definitions that if α is a logical truth or $\alpha \notin K$, then K would keep unchanged after contraction by α , that is, $K - \alpha = K$. And there are two limiting cases of partial meet contraction in which the selection function γ either chooses exactly one element of $K \perp \alpha$, or the whole set $K \perp \alpha$. These two special cases are now known as *maxichoice contraction* and *full meet contraction* respectively ([4]).

Actually, the AGM approach is not only concerned with providing semantic characterizations of belief change but also provides a set of postulates that the contraction operation is required to obey. The main logical accomplishment of AGM is to provide a representation result for these postulates. The notion of partial meet contraction just introduced is complete with respect to the following postulates:

- (K-1) $K - \alpha = Cn(K - \alpha)$ (closure)
- (K-2) $K - \alpha \subseteq K$ (inclusion)
- (K-3) If $\alpha \notin K$, then $K - \alpha = K$. (vacuity)
- (K-4) If $\alpha \notin Cn(\emptyset)$, then $\alpha \notin K - \alpha$ (success)
- (K-5) If $\alpha \equiv \beta$, then $K - \alpha = K - \beta$ (extensionality)
- (K-6) $K \subseteq Cn((K - \alpha) \cup \{\alpha\})$ (recovery)

These six postulates are commonly referred as the basic AGM postulates. All the conditions except possibly *recovery* seem reasonable. In fact, the recovery condition is the most controversial postulate of the foregoing list. There is a relatively large literature on the adequacy of recovery (the following articles are perhaps salient: [15, 11, 16]). Several competing characterizations of contraction that do not obey recovery have been proposed in the literature, for instance, *saturatable contraction* ([11], [7], [6]), *severe withdrawal* ([22]), *systematic withdrawal* ([19]). The purpose of this paper is to offer a complete characterization of saturatable contraction, which was originally proposed by Isaac Levi and later formalized by Hansson and Olsson in [6].

It is possible to strengthen the notion of partial meet contraction by requiring that the selected members of the remainder set are the best elements with respect to an underlying preference relation defined over the elements of the remainder set.

DEFINITION 5. Let K be a theory. A partial meet contraction is relational if and only if it is defined in terms of a selection function γ , and there is a relation \preceq such that for all nonempty $K \perp \alpha$, $\gamma(K \perp \alpha) = \{X \in K \perp \alpha \mid X' \preceq X, \text{ for all } X' \in K \perp \alpha\}$. In particular, if the relation \preceq is transitive, then the partial meet contraction is transitively relational.

This semantic requirement is mirrored by the validity of two additional postulates in addition to the first six basic postulates:

- (K-7) $(K - \alpha) \cap (K - \beta) \subseteq K - (\alpha \wedge \beta)$ (conjunctive overlap)
- (K-8) If $\alpha \notin K - (\alpha \wedge \beta)$, then $K - (\alpha \wedge \beta) \subseteq K - \alpha$ (conjunctive inclusion)

So, the general representation result offered by AGM can now be stated as follows:

THEOREM 6. Let K be a theory. Then an operation $-$ on K is a partial meet contraction if and only if it satisfies the postulates (K-1) to (K-6). Furthermore, $-$ is a transitively relational partial meet contraction if and only if it satisfies the postulates (K-1) to (K-8).

Besides these postulates, there are several relevant postulates which will be used repeatedly in the foregoing sections.

- **Conjunctive factoring:** Either $K - (\alpha \wedge \beta) = K - \alpha$, $K - (\alpha \wedge \beta) = K - \beta$, or $K - (\alpha \wedge \beta) = (K - \alpha) \cap (K - \beta)$.
- **Partial antitony:** $(K - \alpha) \cap Cn(\alpha) \subseteq K - (\alpha \wedge \beta)$.
- **Conjunctive covering:** Either $K - (\alpha \wedge \beta) \subseteq K - \alpha$ or $K - (\alpha \wedge \beta) \subseteq K - \beta$.

All these three postulates are satisfied by the operator of partial meet contraction. And the latter two conditions, i.e., *partial antitony* and *conjunctive covering*, in fact play an important role in establishing the completeness result for partial meet contraction. Moreover, it has been shown that given the basic AGM postulates *conjunctive factoring* is satisfied if and only if *conjunctive overlap* and *conjunctive inclusion* are satisfied.

3. SATURATABLE CONTRACTIONS

In the preceding section we have mentioned that recovery is the most controversial AGM postulate. The basic idea of recovery is that if we first contract a sentence α from a belief set K and then put α back in, the resulting set of beliefs should be precisely the original belief set K . This postulate can be derived from a more general principle requiring that losses of information should be minimized in contraction.

Isaac Levi was one of the first researchers in this area who rejected both recovery and the idea that informational loss should be minimized in contraction. Furthermore, Levi ([11]) proposed two kinds of contraction functions which are now well known as saturatable contraction and mild contraction respectively where recovery fails. Instead of keeping losses of information at a minimum, Levi proposes to use the principle of minimal loss of *informational value* as the main guidance for formulating the concept of contraction. This is one of the salient philosophical motivations behind the proposal of these operations.

Saturatable contraction is the most general operation. Mild contraction is a particular kind of saturatable contraction. A full characterization of mild contraction exists ([2, 13]), but there is no such result characterizing the most general operation of saturatable contraction. The purpose of this paper is to close this gap by offering a complete characterization of saturatable contraction.

We know that the operation of partial meet contraction is defined as selecting elements from the remainder set $K \perp \alpha$. Levi complains that selecting only from this feasible set is unduly restrictive. He proposes instead as a feasible set a larger set of epistemic options. Levi argues that besides the elements of $K \perp \alpha$, there are many other subsets of K that can also guarantee minimal loss of informational value. The common feature of these subsets of K is that when adding $\neg\alpha$ into it, then the logical closure of the resulting set is maximally consistent. Levi therefore proposes that we should extend the choice set from $K \perp \alpha$ into a superset of $K \perp \alpha$ consisting of all the logically closed subsets of K having this common feature. This is what is usually called *saturatable set*, denoted by $S(K, \alpha)$.

DEFINITION 7. *Let K be a theory. Then $X \in S(K, \alpha)$ if and only if it satisfies the following conditions:*

- (i) $X \subseteq K$;
- (ii) $X = Cn(X)$;
- (iii) $Cn(X \cup \{\neg\alpha\})$ is maximally consistent in L .

Of course, it follows from these definitions that saturatable sets contain the corresponding remainder sets:

PROPOSITION 8. *(Hansson, 1999) Let K be a theory. For any sentence $\alpha \in K$, it holds that if $X \in K \perp \alpha$, then $X \in S(K, \alpha)$.*

All the elements of $K \perp \alpha$ belongs to $S(K, \alpha)$, but some elements in $S(K, \alpha)$ could possibly fail to be members of $K \perp \alpha$. This can be easily seen from the following example ([7]). Consider the language L which primitively contains only two logically independent sentences p and q . Let $K = Cn(\{p, q\})$. It is easy to verify that both $Cn(\{q \rightarrow p\})$ and $Cn(\{\neg q \rightarrow p\})$ are elements of $S(K, p)$, but they do not belong to $K \perp p$.

It has been noted already that remainder sets behave very well and have many nice and natural properties. For example, it has been shown that $K \perp(\alpha \wedge \beta) = K \perp \alpha \cup K \perp \beta$ for all $\alpha, \beta \in K$. However, most of the well-behaved properties that are satisfied by remainder sets actually do not hold for saturatable sets. In any event, we can show some analogous properties for saturatable sets, which can serve similar functions in some proofs. The next two

propositions are needed in order to establish a soundness result for saturatable contractions.

PROPOSITION 9. *(Hansson and Olsson, 1995) Let K be a theory. Then for any sentences α and β , $S(K, \alpha \wedge \beta) \subseteq S(K, \alpha) \cup S(K, \beta)$.*

One important claim that should be noted is that if $X \in K \perp \alpha$, then $X \in S(K, \alpha \wedge \beta)$. This claim can be justified by general properties of remainder sets and the previous proposition. It turns out that this property will be quite useful in the following proofs. It is easy to see that the converse subset relationship does not hold for saturatable sets. However, we can show a much weaker property for saturatable sets, which is formally stated in the following proposition.

PROPOSITION 10. *(Hansson and Olsson, 1995) Let α, β , and δ be elements of the logically closed set K . If $\delta \notin Y \in S(K, \alpha)$, then there is some Z such that $Z \subseteq Y \in S(K, \alpha \wedge \beta)$ and $\delta \notin Z$.*

Recall that the operation of partial meet contraction is based on deploying a selection among the elements of $K \perp \alpha$. Under this new setting we can have an analogous characterization of selection function:

DEFINITION 11. *Let K be a theory. A selection function for K is a function γ such that for all sentences α :*

- (i) *If $S(K, \alpha)$ is nonempty, then $\gamma(S(K, \alpha))$ is a nonempty subset of $S(K, \alpha)$,*
- (ii) *If $S(K, \alpha)$ is empty, then $\gamma(S(K, \alpha)) = \{K\}$.*

Now we have enough elements to introduce the most elemental operation proposed in [6] which the authors call *Levi-contraction*:

DEFINITION 12. *(Hansson and Olsson, 1995) Let K be a theory. Then \div is a Levi-contraction for K if and only if there exists a function γ for K such that for all sentences α of the underlying language:*

- (i) *If $\alpha \in K$, then $K \div \alpha = \bigcap \gamma(S(K, \alpha))$, and*
- (ii) *If $\alpha \notin K$, then $K \div \alpha = K$*

Then Hansson and Olsson offer a representation result for this operation in terms of the postulates of closure, inclusion, vacuity, success, extensionality and failure, requiring that $K \div \alpha = K$ when α is a tautology. To compress notation we will use below a modified postulate of vacuity that encompasses the AGM postulate of vacuity and failure.

So, this is the main representation result offered in [6]. It has been pointed out before that the main guiding principle used in articulating saturatable contractions should be the principle of *Cognitive Economy*, which basically says that contraction operations should keep loss of informational value to a minimum. In order to formalize this notion we will introduce a value function V from the power set of the set of formulae into the real numbers. Of course, this function is rather unconstrained. We will assume here only very mild additional conditions on informational value. For instance, we will assume the following constraint:

- **(Weak Monotony)** If $X \subset Y$, then $V(X) \leq V(Y)$.

This is an intuitive principle that makes permissible that two sets carry equal informational value even when one of the sets is strictly larger than the other. The additional information might not be valuable at all and therefore the level of informational value of the larger set might remain equal to the informational value of the smaller set.

A similar idea can be presented in a purely qualitative way via the use of a transitive, weakly monotonic relation among sets of propositions rather than a value function.

DEFINITION 13. Let K be a theory. An operation \div is a relation-based saturatable contraction if and only if it is defined in terms of a selection function γ that is generated from a transitive, weakly monotonic relation \leq on the logically closed subsets of K , such that:

- (i) if $\alpha \in K$, then $K \div \alpha = \bigcap \gamma(S(K, \alpha))$, where $\gamma(S(K, \alpha)) = \{X \in S(K, \alpha) \mid Y \leq X, \text{ for all } Y \in S(K, \alpha)\}$, for $\alpha \in K \setminus Cn(\emptyset)$,⁵
- (ii) if $\alpha \notin K$, then $K \div \alpha = K$.

In contrast with partial meet contraction, the case of vacuous contraction must be separately specified in order to ensure that the postulate of *vacuity* would be satisfied. Unlike remainder sets, when $\alpha \notin K$, it may well be the case that K is not the unique element of $S(K, \alpha)$, and thus perhaps $K \div \alpha = \bigcap \gamma(S(K, \alpha)) \neq K$. Hence without the second clause, the postulate of *vacuity* which Levi also regards to be a reasonable property of contraction would not hold.

Now we can re-introduce a value function as follows:

DEFINITION 14. Let K be a theory. An operation \div is a value-based saturatable contraction if and only if it is defined in terms of a selection function γ that is generated from a weakly monotonic value function V , such that:

- (i) if $\alpha \in K$, then $K \div \alpha = \bigcap \gamma(S(K, \alpha))$, where $\gamma(S(K, \alpha)) = \{X \in S(K, \alpha) \mid V(Y) \leq V(X), \text{ for all } Y \in S(K, \alpha)\}$, for $\alpha \in K \setminus Cn(\emptyset)$;
- (ii) if $\alpha \notin K$, then $K \div \alpha = K$.

This operation was also introduced by Hansson and Olsson in their paper but in this case they only proved the soundness of some postulates rather than a completeness result. Here we will provide a characterization of relation-based saturatable contraction in terms of a set of postulates and we will show that the result is easily extendable to the notion of value-based saturatable contraction in the next section. The axioms presented here are well known in the literature and their names are also more or less standard.

- ($K \div 1$) $K \div \alpha = Cn(K \div \alpha)$. (closure)
- ($K \div 2$) $K \div \alpha \subseteq K$. (inclusion)
- ($K \div 3$) If $\alpha \notin K$ or $\vdash \alpha$, then $K \div \alpha = K$. (vacuity)
- ($K \div 4$) If $\not\vdash \alpha$, then $\alpha \notin K \div \alpha$. (success)
- ($K \div 6$) If $Cn(\alpha) = Cn(\beta)$ then $K \div \alpha = K \div \beta$. (extensionality)
- ($K \div 7$) $(K \div \alpha) \cap (K \div \beta) \subseteq K \div (\alpha \wedge \beta)$. (conjunctive overlap)
- ($K \div 7c$) If $\alpha \in K \div (\alpha \wedge \beta)$, then $K \div \beta \subseteq K \div (\alpha \wedge \beta)$. (conjunctive reduction)
- ($K \div 8$) If $\alpha \notin K \div (\alpha \wedge \beta)$, then $K \div (\alpha \wedge \beta) \subseteq K \div \alpha$. (conjunctive inclusion)

There are two additional conditions that interest us:

- ($K \div 9$) $(K \div \alpha) \cap (K \div \beta) \subseteq K \div (\alpha \wedge \beta)$. (partial antitony)

⁵We thank Arthur Paul Pedersen for pointing out this important qualification. Note that for any $\alpha \in Cn(\emptyset)$ it holds that $S(K, \alpha) = \emptyset$. And it follows from the definition of selection function that $\gamma(S(K, \alpha)) = K$. If the condition on γ were required to hold for $\alpha \in Cn(\emptyset)$, it would imply that $K \in S(K, \alpha)$, which contradicts the condition $S(K, \alpha) = \emptyset$.

- ($K \div 10$) Either $K \div (\alpha \wedge \beta) = K \div \alpha$, $K \div (\alpha \wedge \beta) = K \div \beta$, or $K \div (\alpha \wedge \beta) = (K \div \alpha) \cap (K \div \beta)$. (conjunctive factoring)

Note that all the postulates, except *conjunctive reduction*, *partial antitony*, and *conjunctive factoring*, appear in the standard axiomatization of AGM contraction. Our main goal in this section is to prove a soundness result for these postulates:

THEOREM 15. (Soundness) Let K be a theory and let \div be a relation-based saturatable contraction for K . Then \div satisfies the postulates of *closure*, *inclusion*, *vacuity*, *success*, *extensionality*, *conjunctive overlap*, *conjunctive reduction*, *partial antitony*, and *conjunctive inclusion*.

To prove this fundamental result it is useful to recall a soundness result established by Hansson and Olsson:

LEMMA 16. (Hansson and Olsson, 1995) Let K be a theory and let \div be a relation-based saturatable contraction for K . Then \div satisfies the postulates of *closure*, *inclusion*, *vacuity*, *success*, *extensionality*, *conjunctive overlap*, and *conjunctive inclusion*.

This simplifies our work. And it is worthy pointing out that conjunctive covering mentioned in the previous section is also sound with respect to saturatable contraction, which directly follows from the above lemma and the following proposition.

LEMMA 17. Let K be a theory and let \div be a contraction operator for K that satisfies *vacuity*, *success*, *failure*, and *conjunctive inclusion*. Then it satisfies *conjunctive covering*.

Beside these AGM postulates, we can show that saturatable contraction satisfies conjunctive reduction and conjunctive covering as well.

LEMMA 18. Let K be a theory and let \div be a relation-based saturatable contraction for K . Then it satisfies the postulate of *conjunctive reduction*.

LEMMA 19. Let \div be a relation-based saturatable contraction over K . Then it satisfies the *partial antitony* condition.

The following lemma shows an important connection between conjunctive factoring and conjunctive reduction.

LEMMA 20. Let K be a theory and let \div be a contraction operator for K that satisfies *closure*, *inclusion*, *vacuity*, *extensionality*, and *conjunctive inclusion*. Then it satisfies *conjunctive overlap* and *conjunctive reduction* if and only if it satisfies *conjunctive factoring*.

We have shown the soundness, under the described semantics, of all the AGM postulates other than *recovery*, and also we have proved the soundness of the condition of conjunctive reduction, from which it follows (in the presence of the previous lemma) that the condition of conjunctive factoring is also sound. So, we also have the following result which gives us another useful version of soundness:

THEOREM 21. Let K be a theory and let \div be a relation-based saturatable contraction for K . Then \div satisfies the postulates of *closure*, *inclusion*, *vacuity*, *success*, *extensionality*, *conjunctive inclusion*, *partial antitony* and *conjunctive factoring*.

In Section 5 we will compare the semantics and axioms for saturatable contraction with those for mild contraction, also known as severe withdrawal (following the terminology used by Rott and Pagnucco in [22]).

4. A COMPLETENESS RESULT FOR SATURATABLE CONTRACTIONS

In this section we will present a completeness result for saturatable contraction. Although we define the operation of relation-based saturatable contraction in terms of a qualitative relation \leq , here we are going to show that a quantitative value function V respecting \leq can be defined. This part of proof takes advantage of well-known results in the literature of rational choice.

In order to establish the completeness result, we need to present the notion of completion function which is based on an analogous notion for partial meet contraction introduced in [1]. Given a selection function, we can show that its completion function generates the same saturatable contraction.

DEFINITION 22. *Let K be a theory and γ is a selection function for K . The completion function of γ is the function γ^* such that:*

- (i) $\gamma^*(S(K, \alpha)) = \{X \in S(K, \alpha) \mid \bigcap \gamma(S(K, \alpha)) \subseteq X\}$, if $S(K, \alpha)$ is nonempty;
 - (ii) $\gamma^*(S(K, \alpha)) = \{K\}$, if $S(K, \alpha)$ is empty.
- A selection function γ is completed if and only if $\gamma = \gamma^*$.

In the following proposition, we are going to show that γ^* is also a selection function for K , and determines the same saturatable contraction as γ does. It should be noted that this notion is critical in the formulation of the completeness result.

PROPOSITION 23. *Let K be a theory and let γ be a selection function for K . Then:*

- (i) *The completion function γ^* is a selection function for K ;*
- (ii) $\gamma(S(K, \alpha)) \subseteq \gamma^*(S(K, \alpha))$ for all α ;
- (iii) $\bigcap \gamma(S(K, \alpha)) = \bigcap \gamma^*(S(K, \alpha))$ for all α .

Based upon this property, we now present the central result of this section.

THEOREM 24. *Let K be a theory and let \div be an operator for K that satisfies closure, inclusion, vacuity, success, extensionality, conjunctive inclusion, partial antitony, and conjunctive factoring. Then \div is a relation-based saturatable contraction determined by a selection function γ which satisfies the following conditions:*

- (i) γ satisfies the marking-off identity, i.e., $\gamma(S(K, \alpha)) = \{X \in S(K, \alpha) \mid Y \leq X, \text{ for all } Y \in S(K, \alpha)\}$, for all $\alpha \in K \setminus Cn(\emptyset)$;
- (ii) The relation \leq is transitive;
- (iii) The relation \leq satisfies weak monotonicity, i.e., if $Y \subset X$, then $Y \leq X$.

As was mentioned above, the basic strategy of this proof is similar to the traditional approach in the classical 1985 paper. The basic idea is to define a relation \leq as indicated immediately, and then to show that it satisfies the required properties.

We define a relation \leq over the domain $K\Delta L = \{X \mid X \in S(K, \alpha) \text{ for some } \alpha \in K \setminus Cn(\emptyset)\}$ such that for all $X, Y \in K\Delta L$, $Y \leq X$ if and only if one of the following conditions holds:

- (1) $Y \subseteq X$;
- (2) There is $\alpha \in K \setminus Cn(\emptyset)$ such that $Y \in S(K, \alpha)$ AND There is some $\alpha \in K \setminus Cn(\emptyset)$ [$(X \in S(K, \alpha)$, and $K \div \alpha \subseteq X$), AND for all $\beta \in K \setminus Cn(\emptyset)$ (if $X \subseteq X' \in K \perp (\beta \wedge \alpha)$ and $Y \subseteq Y' \in K \perp (\beta \wedge \alpha)$, and $K \div \beta \subseteq Y'$, then (if $K \div (\alpha \wedge \beta) \neq K \div \alpha$, then $K \div (\alpha \wedge \beta) = K \div \alpha \cap K \div \beta$) and $K \div \beta \subseteq X'$].

Regarding this definition, some explanatory remarks are in order. For $\alpha \in K \setminus Cn(\emptyset)$ and $X \in S(K, \alpha)$, we let X' denote the extension of X belonging to α -remainder set, that is, $X \subseteq X'$

and $X' \in K \perp \alpha$. It is justified to employ such an extension to define the relation since the extension is always uniquely determined. This is easy to show. Suppose by contradiction that X' and X'' are both extensions of $X \in S(K, \alpha)$ in $K \perp \alpha$ and $X' \neq X''$. Then it follows that:

- $Cn(X \cup \{\neg\alpha\}) \subseteq Cn(X' \cup \{\neg\alpha\}) = w'$
- $Cn(X \cup \{\neg\alpha\}) \subseteq Cn(X'' \cup \{\neg\alpha\}) = w''$.

where w' and w'' are $\neg\alpha$ -maximal and consistent sets. w' and w'' should be distinct. To see that it is useful to remember that a member of $K \perp \alpha$ has the canonical form $K \cap w$ where w is a $\neg\alpha$ maximal and consistent set. So, since we assumed that X' and X'' belong to $K \perp \alpha$ and that they are distinct w' and w'' should be distinct. So, since X, X', X'' are all elements of $S(K, \alpha)$, it means that $Cn(X \cup \{\neg\alpha\}) = w$, $Cn(X' \cup \{\neg\alpha\}) = w'$, and $Cn(X'' \cup \{\neg\alpha\}) = w''$ are all maximal consistent sets. Therefore we have that $w \subseteq w'$ and $w \subseteq w''$, where the three sets are maximal and consistent sets and w' and w'' are distinct, which is impossible.

According to the proof, it is clear that the relation \leq is transitive and reflexive, and thus is a *quasi-ordering*. As a matter of terminology, we call a binary relation R^* as an extension of another binary R if and only if it satisfies that (1) $R \subset R^*$, and (2) $\mathcal{P}(R) \subset \mathcal{P}(R^*)$. By utilizing a classical result in the theory of rational choice, we can then obtain an ordering extension of the original relation \leq . More precisely, we can construct an *ordering*, which is reflexive, transitive and complete, based on the previous quasi-ordering \leq . Let us first introduce this important result, which is usually referred as *ordering extension*.

THEOREM 25. (Suzumura, 1983) *A quasi-ordering R , i.e., reflexive and transitive, has an ordering extension R^* , i.e., reflexive, transitive, and complete.*

The proof uses the axiom of choice in the form of Zorn's lemma. Once an ordering relation is obtained, it is easy to show that we can get a quantitative value function by using a central result in the theory of ordinal utility, which is formally presented as follows.

THEOREM 26. (Kreps, 1988) *If X is a countable set, a binary relation \preceq is a preference relation, i.e., complete and transitive, if and only if there exists a function $u: X \rightarrow \mathbb{R}$ such that: for any $x, y \in X$, $x \preceq y$ if and only if $u(x) \leq u(y)$.*

So, via these two results offered above, a numerical value function V can be generated from the qualitative relation \leq defined above. Then it is easy to prove the theorem presented below. This completes the main argument for the completeness result.

THEOREM 27. *Let K be a theory and let \div be an operator for K that satisfies closure, inclusion, vacuity, success, extensionality, conjunctive inclusion, partial antitony, and conjunctive factoring. Then \div is a value-based saturatable contraction determined by a selection function γ which satisfies the following conditions:*

- (i) γ satisfies the marking-off identity, i.e., $\gamma(S(K, \alpha)) = \{X \in S(K, \alpha) \mid V(Y) \leq V(X), \text{ for all } Y \in S(K, \alpha)\}$, for all $\alpha \in K \setminus Cn(\emptyset)$;
- (ii) The value function satisfies weak monotonicity, i.e., if $Y \subset X$, then $V(Y) \leq V(X)$.

4.1 Information Value

We saw in the previous sections that the mere introduction of a notion of epistemic value makes a huge difference in the characterization of contraction. Little has been said though about the notion in question aside from the fact that it obeys Weak Monotony. Levi

was mainly concerned with applications in philosophy of science. So, he conceived the notion in question as establishing a compromise between different dimensions of epistemic value relevant in inquiry (simplicity, predictive or explanatory power, consilience, etc). In other words, he focused on an *overall* notion of epistemic value (similarly one can focus on a notion of *overall similarity* in ontic models of conditionals or one can focus on different aspects of similarity). Nevertheless, Levi did not introduce concrete formal constraints reflecting this intended interpretation. It is an open task to do so. For example one can focus only on simplicity. Are there interesting formal properties reflecting this concrete dimension of the notion of epistemic value? A model where additional constraints are introduced on an overall notion of epistemic value is discussed in the coming section (this is the model of *mild contraction* discussed in [13]).

The central role played by epistemic value in the study of belief change is to introduce explicitly a notion of *epistemic or cognitive utility* in the models of change operators. This move permits a direct connection with decision theoretic ideas: rationality is understood as an exercise in maximization (minimization) of expected epistemic utility (concretely rational contraction requires the minimization of losses of epistemic utility).

5. COMPARISONS AND DISCUSSION

Meyer et al. consider in [19] an operation of *principled withdrawal* (following a terminology for removal operations first proposed by David Makinson in [15]). A removal operation is a principled withdrawal if and only if it obeys all the AGM axioms for contraction with the exception of recovery. There are various forms of principled withdrawal. The operation of saturatable contraction characterized above is obviously a principled withdrawal. But there are many and relatively well known operations of this type. Perhaps one that is salient is the operation that Rott and Pagnucco call *severe withdrawal*, known otherwise as *mild contraction* (this second terminology is due to Levi who presumably changes terminology to indicate that what might look severe from the point of view of pure informational loss might not look this way if one changes perspective and focuses on information value).

There are many ways of motivating severe withdrawals. But for us the simpler way of introducing them is as a species of saturatable contractions where the value function is further constrained. This is the strategy used in [2] to introduce them. The basic idea is to impose the following constraint on value functions:

(Weak Min) For any finite $T \subset S(K, \alpha)$, $V(\bigcap T) = \min(V(X) : X \in T)$.

In general for any two potential contractions X and Y the value of their intersection is the minimum of the values of X and Y . In [2] these principles are derived from more primitive axioms in an attempt to justify them in general (see the principles of Weak Monotony, Extended Weak Monotony and Weak Intersection Equality presented in section 2 of [2]). [2] presents an argument showing that saturatable contractions obeying these extra axioms are complete with respect to the following syntax.⁶

$(K \approx 1)$ $K \approx \alpha = Cn(K \approx \alpha)$. (closure)

$(K \approx 2)$ $K \approx \alpha \subseteq K$. (inclusion)

$(K \approx 3)$ If $\alpha \notin K$ or $\vdash \alpha$, then $K \approx \alpha = K$. (vacuity)

⁶The axiom system presented in [2] substitutes $\alpha \in Cn(LK)$ for $\vdash \alpha$, where LK is a basic theory included in K .

$(K \approx 4)$ If $\not\vdash \alpha$, then $\alpha \notin K \approx \alpha$. (success)

$(K \approx 5)$ If $Cn(\alpha) = Cn(\beta)$ then $K \approx \alpha = K \approx \beta$. (extensionality)

$(K \approx 6)$ If $\vdash \alpha$ then $K \subseteq K \approx \alpha$. (failure)

$(K \approx 7)$ If $\not\vdash \alpha$, then $K \approx \alpha \subseteq K \approx (\alpha \wedge \beta)$. (antitony)

$(K \approx 8)$ If $\alpha \notin K \approx (\alpha \wedge \beta)$, then $K \approx (\alpha \wedge \beta) \subseteq K \approx \alpha$. (conjunctive inclusion)

Rott and Pagnucco offer in [22] an independent representation result for the same syntax in terms of plausibility models (using Grove spheres). And they also offer a general philosophical argument defending the coherence of this particular form of contraction.

Some of the theorems of severe withdrawals are nevertheless puzzling. For example, one can derive that either $K \approx \phi \subseteq K \approx \psi$ or $K \approx \psi \subseteq K \approx \phi$. Severe withdrawals are too orderly: any two arbitrary contractions of an arbitrary theory are such that either one of them entails the other or vice versa. This seems too strong although it is not difficult to see that this is a trivial consequence of the semantics used by Rott and Pagnucco or the systems of shells of information value used in [2]. The axiom of antitony has been criticized as well. For example, Hansson says in passing in his book ([7]) that a full version of antitony (without the proviso that the contracted sentence α is not a logical theorem) ‘does not hold for any sensible notion of contraction’ (see page 117 of [7]).⁷

There are other theorems of mild contractions that seems rather unintuitive like the one called *expulsiveness* by Hansson. The postulate requires for any two non-tautological sentences α and β (alternatively for any sentences α, β not included in $LK \subseteq K$) that either $\alpha \notin K \approx \beta$ or $\beta \notin K \approx \alpha$. Hansson argues as follows against this condition ([8]):

This is a highly implausible property of belief contraction, since it does not allow unrelated beliefs to be undisturbed by each other’s contraction. Consider a scholar who believes that her car is parked in front of the house. She also believes that Shakespeare wrote the Tempest. It should be possible for her to give up the first of these beliefs while retaining the second. She should also be able to give up the second without giving up the first. Expulsiveness does not allow this. The construction of a plausible operation of contraction for belief sets that does not satisfy Recovery is still an open issue.

Notice that expulsiveness is implausible also for *mutually related* beliefs. Consider the example when the two relevant beliefs are that

⁷Levi has defended antitony by appealing to the use of partitions in the presentation of contraction. Much of the counterexamples to antitony appeal to cases where the sentences α and β used in the postulate are mutually irrelevant. The use of partitions permits to filter irrelevant cases, in the sense that the two sentences in question are potential answers to the same issue. One can certainly use a semantics where partitions of this sort are deployed. In [2] such a semantics is used. But in [2] a complete axiomatization is presented from which the axioms discussed here are derivable. In particular the axiom we are discussing here is derivable for any sentence α, β , without any further syntactic restrictions. We are considering here the adequacy of axioms independently of the semantics utilized to validate them (the possible world semantics of Rott and Pagnucco, Levi’s partitional semantics, etc). But even if one only considers instances of this axiom where the two sentences are potential answers to the same issue, the requirement that any two representable arbitrary contractions obey this tidy entailment pattern seems too orderly to be true.

her car is parked in front of the house and that the car contains a bomb. It seems that it should be plausible to give up the belief that the car is parked in front of her house but preserve the belief that the car contains a bomb. It also seems perfectly possible to give up that the car contains a bomb while preserving the belief that the car is parked in front of the scholar's house.

Many ([14], [19]) have seen severe withdrawals and AGM contraction as two limit cases of a reasonable operation. The idea is presented in [14] and later labelled the *interpolation thesis* by Rott (see, for example, [21]). If we use \approx to denote severe withdrawal and \div to denote AGM contraction, the interpolation thesis states that we should regard as reasonable only those withdrawals \sim for which $K \approx \alpha \subseteq K \sim \alpha \subseteq K \div \alpha$, for any sentence α in the language.

It is not difficult to see that saturable contraction is a reasonable withdrawal. But not all reasonable withdrawals are saturable contractions. For Meyer showed in his thesis [18] that there is a reasonable withdrawal that does not satisfy the conjunctive overlap and conjunctive inclusion postulates of AGM contraction, which saturable contraction certainly satisfies.

The debate as to what is the most reasonable notion of contraction is perhaps still open. One moral of our analysis here is that saturable contraction should be certainly among the best contenders in this competition. Saturable contraction has a decision theoretic foundation deploying a very weak notion of value. It is for sure a principled withdrawal. And the properties that characterize it do not contain any axiom that one would like to criticize with the possible exception of the seventh postulate that has been criticized recently (see [3] and [23]). But these criticisms seem to apply in contexts where one appeals to social norms of some sort. Aside from these particular cases it seems that saturable contraction is a rather well behaved notion of contraction. Our main goal in this note was to study its logical properties rather than defend it via philosophical arguments. But the completeness result itself contributes to this debate by making explicit the syntactic commitments that one contracts when one endorses it semantically. And, as we explained above, these commitments do not seem to include particularly problematic axioms while preserving much of the logical strength of operations like AGM or severe withdrawal (which do entail prima facie problematic axioms).

Due to space limitation we do not include here all the proofs but the interested reader can download the entire paper with complete proofs in the following web site: <http://www.hss.cmu.edu/philosophy/faculty-arlocosta.php>. We do provide nevertheless a definition of the relation used in the main completeness result and the proof of the central theorem (Theorem 24). The reader familiar with the classical AGM paper can appreciate that this definition preserves the flavor of the proof strategy used in [1] while at the same time some important modifications are introduced to deal with saturable sets (rather than remainder sets). We show that much of the AGM original approach can be used in order to characterize a well behaved notion of contraction not constrained by Recovery. Of course it is possible to construct proofs that are less syntax-dependent but we propose this and other extensions of the results presented here as future work.

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pointed out to problems that lead to improvements and a more compact formulations of the main argument.

Appendix

Theorem 24. Let K be a theory and let \div be an operator for K that satisfies closure, inclusion, vacuity, success, extensionality, conjunctive inclusion, partial antitony and conjunctive factoring. Then \div is a relation-based saturable contraction determined by a selection function γ which satisfies the following conditions:

- (i) γ satisfies the marking-off identity, i.e., $\gamma(S(K, \alpha)) = \{X \in S(K, \alpha) \mid Y \leq X, \text{ for all } Y \in S(K, \alpha)\}$, for all $\alpha \in K \setminus Cn(\emptyset)$;
- (ii) The relation \leq is transitive;
- (iii) The relation \leq satisfies the weak monotonicity, i.e., if $Y \subseteq X$, then $Y \leq X$.

PROOF. Let K be any theory, and \div a Levi-contraction function over K , determined by a selection function γ . We have to show that if \div satisfies in addition *conjunctive inclusion*, *partial antitony*, and *conjunctive factoring*, then there is a transitive and weakly monotonic relation \leq such that γ satisfies the marking-off identity with respect to this relation. It is easy to verify that the selection function γ in question is complete. So let us define explicitly the relation \leq over the domain $K\Delta L = \{X \mid X \in S(K, \alpha) \text{ for some } \alpha \in K \setminus Cn(\emptyset)\}$ such that for all $X, Y \in K\Delta L$, $Y \leq X$ if and only if one of the following conditions holds:

- (1) $Y \subseteq X$;
- (2) There is $\alpha \in K \setminus Cn(\emptyset)$ such that $Y \in S(K, \alpha)$ AND There is some $\alpha \in K \setminus Cn(\emptyset)$ [$X \in S(K, \alpha)$, and $K \div \alpha \subseteq X$], AND for all $\beta \in K \setminus Cn(\emptyset)$ (if $X \subseteq X' \in K \perp (\beta \wedge \alpha)$ and $Y \subseteq Y' \in K \perp (\beta \wedge \alpha)$, and $K \div \beta \subseteq Y'$, then (if $K \div (\alpha \wedge \beta) \neq K \div \alpha$, then $K \div (\alpha \wedge \beta) = K \div \alpha \cap K \div \beta$) and $K \div \beta \subseteq X'$]).

We need to show that the relation defined in this way satisfies the three conditions given in the proposition.

(i) We first show that γ satisfies the marking-off identity. Let $\alpha \in K \setminus Cn(\emptyset)$. Suppose that $X \in \gamma(S(K, \alpha)) \subseteq S(K, \alpha)$. We need to show that $X \in \{X \in S(K, \alpha) \mid Y \leq X, \text{ for all } Y \in S(K, \alpha)\}$. It suffices to show that $Y \leq X$ for any $Y \in S(K, \alpha)$.

Then it is clear that the first part of condition (2) is already satisfied. Let $\beta \in K \setminus Cn(\emptyset)$. And suppose that $K \div \beta \subseteq Y'$. We need to show that $K \div \beta \subseteq X'$ and, if $K \div \alpha \wedge \beta \neq K \div \alpha$, then $K \div (\alpha \wedge \beta) = K \div \alpha \cap K \div \beta$.

Let us focus first on $K \div (\alpha \wedge \beta)$. We know by *conjunctive covering* that either $K \div (\alpha \wedge \beta) \subseteq K \div \beta$ or $K \div (\alpha \wedge \beta) \subseteq K \div \alpha$. Assume that $K \div (\alpha \wedge \beta) \subseteq K \div \beta$. Since it is assumed $K \div \beta \subseteq Y'$, we have $K \div (\alpha \wedge \beta) \subseteq K \div \beta \subseteq Y' \in K \perp \alpha$, which implies that $\alpha \notin K \div (\alpha \wedge \beta)$. Thus by *conjunctive inclusion* we have that $K \div (\alpha \wedge \beta) \subseteq K \div \alpha$. Therefore in both cases we have that $K \div (\alpha \wedge \beta) \subseteq K \div \alpha$.

By *conjunctive factoring*, $K \div (\alpha \wedge \beta)$ is either equal to $K \div \alpha$, to $K \div \beta$ or to the intersection of these two sets. Suppose that $K \div (\alpha \wedge \beta) \neq K \div \alpha$. If $K \div (\alpha \wedge \beta) = K \div \beta$, then it implies that $K \div (\alpha \wedge \beta) = K \div \alpha \cap K \div \beta$, for we already have that $K \div (\alpha \wedge \beta) \subseteq K \div \alpha$. Hence it holds that $K \div (\alpha \wedge \beta) = K \div \alpha \cap K \div \beta$. This proves that if $K \div (\alpha \wedge \beta) \neq K \div \alpha$, then $K \div (\alpha \wedge \beta) = K \div \alpha \cap K \div \beta$.

We need to prove now that $K \div \beta \subseteq X'$. Assume that $\delta \in K \div \beta$. Consider the following instance of *partial antitony*:

$$(K \div \beta) \cap Cn(\beta) \subseteq K \div (\alpha \wedge \beta).$$

Since we assumed that $\delta \in K \div \beta$, we have as well that $\delta \vee \beta \vee \alpha \in K \div \beta$, and since $\delta \vee \beta \vee \alpha \in Cn(\beta)$, we have that $\delta \vee \beta \vee \alpha \in K \div (\alpha \wedge \beta)$. Moreover since we have that $K \div (\alpha \wedge \beta) \subseteq K \div \alpha \subseteq X \subseteq X'$, we have that $\delta \vee \beta \vee \alpha \in X'$.

Note that $X' \in K \perp (\beta \wedge \alpha)$ and $\delta \in K$. It thus implies that $X' \cup \{\alpha, \beta\} \vdash \delta$. So we have that $\neg \beta \rightarrow (\neg \alpha \rightarrow \delta) \in X'$ and $\beta \rightarrow (\alpha \rightarrow \delta) \in X'$, which entail that $\delta \in X'$. It follows that $K \div \beta \subseteq X'$. This completes the proof by establishing the desired result that $Y \leq X$.

For the converse, suppose that $X \notin \gamma(S(K, \alpha))$ for some $\alpha \in K \setminus Cn(\emptyset)$. We need to show that $X \notin \{X \in S(K, \alpha) \mid Y \leq X, \text{ for all } X \in S(K, \alpha)\}$. Note that if $X \notin S(K, \alpha)$, then it trivially follows that $X \notin \{X \in S(K, \alpha) \mid Y \leq X, \text{ for all } X \in S(K, \alpha)\}$. Thus, suppose that $X \in S(K, \alpha)$. To show that $X \notin \{X \in S(K, \alpha) \mid Y \leq X, \text{ for all } X \in S(K, \alpha)\}$, it suffices to find some $Y \in S(K, \alpha)$ such that $Y \not\leq X$. Since $X \in S(K, \alpha)$, it means that $S(K, \alpha)$ is nonempty, which follows from the property of selection function that $\gamma(S(K, \alpha))$ is nonempty. Thus let $Y \in \gamma(S(K, \alpha))$. Then we have that $K \div \alpha = \bigcap \gamma(S(K, \alpha)) \subseteq Y$. On the other hand, since $X \notin \gamma(S(K, \alpha))$, we can conclude from the property of selection function that $\bigcap \gamma(S(K, \alpha)) \not\subseteq X$. We thus have $K \div \alpha \not\subseteq X$. From these, we can conclude that $Y \not\leq X$, which establishes that condition (1) does not hold. And note that $X, Y \in S(K, \alpha)$, $K \div \alpha \subseteq Y$, but $K \div \alpha \not\subseteq X$.

We have to check that condition (2) is not satisfied either. Notice that the negation of condition (2) is given as follows:

\neg There is $\alpha \in K \setminus Cn(\emptyset)$ such that $Y \in S(K, \alpha)$ OR For all $\alpha \in K \setminus Cn(\emptyset)$ [if $(X \in S(K, \alpha), \text{ and } K \div \alpha \subseteq X)$, THEN there is $\beta \in K \setminus Cn(\emptyset)$ ($X \subseteq X' \in K \perp (\beta \wedge \alpha)$ and $Y \subseteq Y' \in K \perp (\beta \wedge \alpha)$, and $K \div \beta \subseteq Y'$, and either \neg (if $K \div (\alpha \wedge \beta) \neq K \div \alpha$, then $K \div (\alpha \wedge \beta) = K \div \alpha \cap K \div \beta$) or $K \div \beta \not\subseteq X'$].

We need to find an adequate β in K in order to prove the truth of the second disjunct. Can we argue that such a β exists in K ? The answer is yes if the belief sets used in the theory are finitely axiomatizable in the sense that a formula ψ represents a belief set K if and only if $K = \{\phi : \psi \vdash \phi\}$. If we assume that all belief sets are representable by a formula in the language we can argue as follows to identify the needed β in K :

Let w be the maximal and consistent belief set such that $Cn(Y \cup \{\neg \alpha\}) = w$. Let $\neg \psi$ represent w . Then clearly $\psi \in K$. Set $\beta = \psi$. Clearly also in this case $K \div \beta \subseteq Y'$. Assume now by contradiction that $K \div \beta \not\subseteq X'$. Let $[K \div \beta]$ be the set of worlds that correspond to the theory $K \div \beta$. It is clear that w is the only $\neg \alpha$ world in $[K \div \beta]$. Now notice that since $X' \neq Y'$ we have that $[Y'] = \{w\} \cup [K]$ and $[Y'] = \{w^*\} \cup [K]$ where w and w^* should be distinct $\neg \alpha$ worlds. But from our assumption we have that $[X'] \subseteq [K \div \beta]$, which entails that w^* is in $[K \div \beta]$. But this is impossible because we already saw that $[K \div \beta]$ should contain a unique $\neg \alpha$ world, namely w .

In addition, we have that $X', Y' \in K \perp (\alpha \wedge \beta)$. Notice that $X, Y \in S(K, \alpha)$. Therefore, $X', Y' \in K \perp \alpha$, and since $K \perp \alpha \subseteq K \perp (\alpha \wedge \beta)$ we are done.

(ii) Next we are going to show that \leq is transitive. Let $X, Y, Z \in K \Delta L$. Suppose that $Z \leq Y$ and $Y \leq X$. We need to show that $Z \leq X$. Let us consider the following cases.

Case 1: Both $Z \leq Y$ and $Y \leq X$ hold due to condition (1). In this case, we have that $Z \subseteq Y$ and $Y \subseteq X$. Thus according to the definition, we have that $Z \subseteq X$ as required.

Case 2: $Z \leq Y$ holds because of condition (1), and $Y \leq X$ holds due to condition (2). So, we have that:

(2') There is $\alpha \in K \setminus Cn(\emptyset)$ such that $Y \in S(K, \alpha)$ AND There is some $\alpha' \in K \setminus Cn(\emptyset)$ [$(X \in S(K, \alpha')$, and $K \div \alpha' \subseteq X)$, AND for all $\beta \in K \setminus Cn(\emptyset)$ (if $X \subseteq X' \in K \perp (\beta \wedge \alpha')$ and $Y \subseteq Y' \in K \perp (\beta \wedge \alpha')$, and $K \div \beta \subseteq Y'$, then (if $K \div (\alpha' \wedge \beta) \neq K \div \alpha'$, then $K \div (\alpha' \wedge \beta) = K \div \alpha' \cap K \div \beta$) and $K \div \beta \subseteq X'$].

And we need to prove that:

(2'') There is $\alpha \in K \setminus Cn(\emptyset)$ such that $Z \in S(K, \alpha)$ AND There is some $\alpha' \in K \setminus Cn(\emptyset)$ [$(X \in S(K, \alpha')$, and $K \div \alpha' \subseteq X)$, AND for all $\beta \in K \setminus Cn(\emptyset)$ (if $X \subseteq X' \in K \perp (\beta \wedge \alpha')$ and $Z \subseteq Z' \in K \perp (\beta \wedge \alpha')$, and $K \div \beta \subseteq Z'$, then (if $K \div (\alpha' \wedge \beta) \neq K \div \alpha'$, then $K \div (\alpha' \wedge \beta) = K \div \alpha' \cap K \div \beta$) and $K \div \beta \subseteq X'$].

First since $Z \leq Y$ holds because of condition (1), we have that $Z \subseteq Y$. And note that $Z, Y \in K \Delta L$. It thus must be the case that $Z' = Y'$ and $Z', Y' \in K \perp \alpha$ for some $\alpha \in K \setminus Cn(\emptyset)$. Consider this particular α . It thus follows from $Z', Y' \in K \perp \alpha$ that $Z, Y \in S(K, \alpha)$ for this particular α . So the first existential clause is satisfied. The second existential clause is also satisfied given that $Y \leq X$ holds due to condition (2). And the universal clause can be easily established for $Z' = Y'$ and the fact that $Y \leq X$ holds due to condition (2).

We are first going to show that $K \div \beta \subseteq X'$. Let $\delta \in K \div \beta$. First it follows from $X' \in K \perp (\beta \wedge \alpha')$, $Z' \in K \perp (\beta \wedge \alpha')$, and $Y' \in K \perp \alpha$ that $X', Y', Z' \in K \perp (\alpha \wedge \beta \wedge \alpha')$. Since $Y \leq X$ holds because of condition (2), we can infer that $K \div (\alpha \wedge \beta) \subseteq X'$ by taking $(\alpha \wedge \beta)$ as the instance of β in (2'). It follows from $\delta \in K \div \beta$ that $\delta \vee \beta \vee \alpha' \in K \div \beta$. And clearly $\delta \vee \beta \vee \alpha' \in Cn(\beta)$. Now consider the following instance of *partial antitony*:

$$(K \div \beta) \cap Cn(\beta) \subseteq K \div (\alpha \wedge \beta).$$

Then it implies that $\delta \vee \beta \vee \alpha' \in K \div (\alpha \wedge \beta) \subseteq X'$. Moreover, since we assumed that $X' \in K \perp (\beta \wedge \alpha')$ and $\delta \in K \div \beta \subseteq K$, we have as well that $X' \cup \{\alpha', \beta\} \vdash \delta$. So these imply that $\delta \in X'$. This completes the proof of $K \div \beta \subseteq X'$.

Case 3: $Z \leq Y$ holds because of condition (2), and $Y \leq X$ holds due to condition (1). This case is similar to the previous case.

Case 4: Both $Z \leq Y$ and $Y \leq X$ hold because condition (2). Unpacking definitions we have:

(2-Z-Y) There is $\alpha \in K \setminus Cn(\emptyset)$ such that $Z \in S(K, \alpha)$ AND There is some $\alpha' \in K \setminus Cn(\emptyset)$ [$(Y \in S(K, \alpha')$, and $K \div \alpha' \subseteq Y)$, AND for all $\beta \in K \setminus Cn(\emptyset)$ (if $Z \subseteq Z' \in K \perp (\beta \wedge \alpha')$ and $Y \subseteq Y' \in K \perp (\beta \wedge \alpha')$, and $K \div \beta \subseteq Y'$, then (if $K \div (\alpha' \wedge \beta) \neq K \div \alpha'$, then $K \div (\alpha' \wedge \beta) = K \div \alpha' \cap K \div \beta$) and $K \div \beta \subseteq Y'$].

On the other hand we also have:

(2-Y-X) There is $\alpha' \in K \setminus Cn(\emptyset)$ such that $Y \in S(K, \alpha')$ AND There is some $\alpha'' \in K \setminus Cn(\emptyset)$ [$(X \in S(K, \alpha'')$, and $K \div \alpha'' \subseteq X)$, AND for all $\beta \in K \setminus Cn(\emptyset)$ (if $X \subseteq X' \in K \perp (\beta \wedge \alpha'')$ and $Y \subseteq Y' \in K \perp (\beta \wedge \alpha'')$, and $K \div \beta \subseteq Y'$, then (if $K \div (\alpha'' \wedge \beta) \neq K \div \alpha''$, then $K \div (\alpha'' \wedge \beta) = K \div \alpha'' \cap K \div \beta$) and $K \div \beta \subseteq X'$].

And we need to prove:

(2-Z-X) There is $\alpha \in K \setminus Cn(\emptyset)$ such that $Z \in S(K, \alpha)$ AND There is some $\alpha'' \in K \setminus Cn(\emptyset)$ [$(X \in S(K, \alpha'')$, and $K \div \alpha'' \subseteq X)$, AND for all $\beta \in K \setminus Cn(\emptyset)$ (if $X \subseteq X' \in K \perp (\beta \wedge \alpha'')$ and $Z \subseteq Z' \in K \perp (\beta \wedge \alpha'')$, and $K \div \beta \subseteq Z'$, then (if $K \div (\alpha'' \wedge \beta) \neq K \div \alpha''$, then $K \div (\alpha'' \wedge \beta) = K \div \alpha'' \cap K \div \beta$) and $K \div \beta \subseteq X'$].

It is clear that the first existential clause of (2-Z-X) is given by (2-Z-Y) while the second is given by (2-Y-X). We need to prove the universally quantified clause. Assume that $X \subseteq X' \in K \perp \beta \wedge \alpha''$ and $Z \subseteq Z' \in K \perp \beta \wedge \alpha''$, and $K \div \beta \subseteq Z'$. We need to prove that $K \div \beta \subseteq X'$ and (if $K \div \alpha'' \wedge \beta \neq K \div \alpha''$, then $K \div (\alpha'' \wedge \beta) = K \div \alpha'' \cap K \div \beta$). We will prove first that $K \div \beta \subseteq X'$, and then we will use this result to prove that (if $K \div \alpha'' \wedge \beta \neq K \div \alpha''$, then $K \div (\alpha'' \wedge \beta) = K \div \alpha'' \cap K \div \beta$).

By the property of reminder sets establishing that $K \perp (\beta \wedge \alpha' \wedge \alpha'') = K \perp (\beta \wedge \alpha'') \cup K \perp \alpha'$, we have that $Z', X', Y' \in K \perp (\beta \wedge \alpha' \wedge \alpha'')$. In fact, $Y \in S(K, \alpha')$, and thus $Y' \in K \perp \alpha'$ and Z', X' are in $K \perp (\beta \wedge \alpha'')$.

By *conjunctive covering*, either $K \div (\beta \wedge \alpha' \wedge \alpha'') \subseteq K \div \beta$ or $K \div (\beta \wedge \alpha' \wedge \alpha'') \subseteq K \div (\alpha' \wedge \alpha'')$. In the first case we have that $K \div (\beta \wedge \alpha' \wedge \alpha'') \subseteq Z'$. So, by taking appropriate instances of β in (2-Z-Y) and (2-Y-X) we have that $K \div (\beta \wedge \alpha' \wedge \alpha'') \subseteq X'$.

On the other hand, if we consider the case $K \div (\beta \wedge \alpha' \wedge \alpha'') \subseteq K \div (\alpha' \wedge \alpha'')$, we proceed by cases. By *conjunctive covering*, either $K \div (\alpha' \wedge \alpha'') \subseteq K \div \alpha'$ or $K \div (\alpha' \wedge \alpha'') \subseteq K \div \alpha''$. In the first case we have that $K \div (\beta \wedge \alpha' \wedge \alpha'') \subseteq K \div \alpha' \subseteq Y'$. Then we can fire (2-Y-X) and we get $K \div (\beta \wedge \alpha' \wedge \alpha'') \subseteq X'$. In the second case we have that $K \div (\beta \wedge \alpha' \wedge \alpha'') \subseteq K \div \alpha'' \subseteq X'$. So, in all possible cases we have that $K \div (\beta \wedge \alpha' \wedge \alpha'') \subseteq X'$.

We need to prove that $K \div \beta \subseteq X'$. Assume that $\delta \in K \div \beta$. Consider the following instance of *partial antitony*:

$$(K \div \beta) \cap Cn(\beta) \subseteq K \div (\beta \wedge \alpha' \wedge \alpha'').$$

Since we assumed that $\delta \in K \div \beta$ we have as well that $(\beta \vee \alpha' \vee \alpha'' \vee \delta) \in K \div \beta$, and since $(\beta \vee \alpha' \vee \alpha'' \vee \delta) \in Cn(\beta)$ we have that $(\beta \vee \alpha' \vee \alpha'' \vee \delta) \in K \div (\beta \wedge \alpha' \wedge \alpha'') \subseteq X'$.

On the other hand, since $X \in K \perp (\alpha'' \wedge \beta)$ and $\delta \in K$ we have that $X' \cup \{\alpha'', \beta\} \vdash \delta$ from which (by monotonicity of the notion of consequence) we have: $X' \cup \{\alpha'', \alpha', \beta\} \vdash \delta$. So, clearly $\delta \in X'$. This completes the proof of $K \div \beta \subseteq X'$.

Now we need to prove that (if $K \div (\alpha'' \wedge \beta) \neq K \div \alpha''$, then $K \div (\alpha'' \wedge \beta) = K \div \alpha'' \cap K \div \beta$). Assume the antecedent. By *conjunctive covering* either $K \div (\alpha'' \wedge \beta) \subseteq K \div \beta$ or $K \div (\alpha'' \wedge \beta) \subseteq K \div \alpha''$. Let's start with the first disjunct. Since we have that $K \div (\alpha'' \wedge \beta) \subseteq K \div \beta$ and we proved that $K \div \beta \subseteq X' \subseteq K \perp \alpha''$, we know that $\alpha'' \notin K \div (\alpha'' \wedge \beta)$. Therefore by *conjunctive inclusion* $K \div (\alpha'' \wedge \beta) \subseteq K \div \alpha''$. In addition by *conjunctive overlap* we have that $K \div \alpha'' \cap K \div \beta \subseteq K \div (\alpha'' \wedge \beta)$. Therefore we have that $K \div \alpha'' \cap K \div \beta = K \div (\alpha'' \wedge \beta)$ as desired.

Let us consider now the second disjunct $K \div (\alpha'' \wedge \beta) \subseteq K \div \alpha''$. Since we assumed that $K \div (\alpha'' \wedge \beta) \neq K \div \alpha''$, we know that $K \div \alpha'' \not\subseteq K \div (\alpha'' \wedge \beta)$. Therefore by *conjunctive reduction* we have that $\beta \notin K \div (\alpha'' \wedge \beta)$, and this by *conjunctive inclusion* yields that $K \div (\alpha'' \wedge \beta) \subseteq K \div \beta$. Now by invoking *conjunctive overlap* again, we have that $K \div \alpha'' \cap K \div \beta = K \div (\alpha'' \wedge \beta)$ as desired. This completes the general proof.

(iii) We need to show that \leq satisfies the weak monotonicity. Let $X, Y \in S(K, \alpha)$ for some $\alpha \in K \setminus Cn(\emptyset)$. Assume that $Y \subset X$. According to condition (1), it directly follows from $Y \subset X$ that $Y \leq X$ as desired. \square

7. REFERENCES

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