Equivalence of the Information Structure with Unawareness to the Logic of Awareness

[Extended Abstract] *

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ABSTRACT

This paper proves the Li (2009) unawareness structure equivalent to the single-agent propositionally generated logic of awareness of Fagin and Halpern (1988). For any model of one type one can construct a model of the other type describing the same belief and awareness. Li starts from an agent unable to perceive aspects of the world and distinguish states, modelled with subjective state spaces coarser than the objective state space. Fagin and Halpern limit the agent's language or cognitive ability to reasoning only about a subset of the propositions describing the world. Equivalence of these approaches suggests they capture a natural notion of unawareness in a minimal way.

Categories and Subject Descriptors

F.4.1 [**Mathematical logic and formal languages**]: Modal logic

General Terms

Theory

Keywords

awareness, unawareness, knowledge, epistemology

1. INTRODUCTION

Economic models describe at least three types of uncertainty—risk, ambiguity and unawareness. In risk, the possible outcomes and their probabilities are known, such as in the case of a coin toss. With ambiguity, the outcomes are still known, but there is no obvious single probability distribution, for example tomorrow can be sunny or cloudy, but it is difficult to find one objective probability for these outcomes. In the case of ambiguity there is instead a set of possible probabilities or a probability-like structure. Under unawareness, the possible realizations of uncertainty are not known, e.g. which new viruses could be discovered in the next ten years.

In the case of all types of uncertainty, the agent does not have full first-order knowledge. The difference is in the extent of higher order knowledge. Under risk and ambiguity, the agent knows *that* he does not know and knows *what* he does not know. In the case of unawareness, the agent knows neither. An intermediate concept is awareness of unawareness, where the agent knows *that* he does not know, but does not know *what* he does not know. A person may know there exist viruses he is unaware of, but not know what the possible viruses are.

The logic literature has used unawareness as a way around the unrealistic property of logical omniscience (the agent knows all the logical consequences of his knowledge). A common example in the literature is from A. C. Doyle's story "Silver Blaze" where Watson fails to deduce the impossibility of an intruder in the stable from the fact that the dog did not bark. The facts were available to Watson, but he was unaware of the question of the intruder until told by Holmes. To model real agents, logical omniscience must be relaxed.

One way to restrict logical deductions is to use syntactic belief operators unaffected by the logical connections between formulas, so the agent believes certain formulas by definition. Syntactic belief operators are the most general approach, but close to assuming the result. Another approach is to allow the agent to think about inconsistent states where logically contradictory statements may be true. A third approach is to make the agent unaware of some formulas and require awareness for belief. The agent cannot believe some formulas implied by already believed formulas, due to unawareness. Some forms of logical boundedness, e.g. computational limitations preventing the proving of a known conclusion, are difficult to interpret as unawareness.

Representing unawareness in the set-based models used in economics is difficult. [3] proved that in a standard state-space model with operators for knowledge and unawareness satisfying some natural axioms, the agent is unaware of everything or nothing. The solution suggested by [3] is to separate the objective and subjective state spaces (the modeller's and the agent's views of the environment).

There are a number of ways in which the literature has circumvented the [3] impossibility result. The unawareness structure constructed by [11] (henceforth the Li structure)

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has the agent use a (possibly different) subjective state space at each state. The unawareness operator expresses the agent's incomplete subjective description of relevant aspects of the world. The logic of awareness and Kripke structure of [4] (subsequently called the FH structure) describe nontrivial unawareness by making the agent aware of only a subset of the propositions in the model. The agent can believe or disbelieve only formulas constructed from these propositions. The approach of [4] has been further extended by [7] to incorporate quantification and to enable the agent to reason about the existence of unawareness. A similar approach to [4] is taken in the logic defined by [13]—the agent can reason only about the atomic sentences of which he is aware. [8] introduce a lattice of state spaces with different expressive power. The agent's possibility set at one state is contained in a possibly less expressive state space, which represents the incomplete awareness in the original state. Early models without logical omniscience are [10] and [14], which allow the agent to consider possible inconsistent states (called impossible worlds). Impossible worlds models can also be used to describe unawareness as in [18]. These have been augmented with quantification by [16, 17] to express awareness of unawareness.

The concept of unawareness used in these models expresses a relationship between different sets (state spaces or languages of formulas). This differs from [3] where the unawareness operator is confined to a single set, so it is not surprising the models are unaffected by the impossibility result.

Some of the models capturing nontrivial unawareness have been shown to be equivalent to each other. The notion of equivalence used in the previous literature and this paper is that for each model of one type we can construct a model of the other type and vice versa so that a formula is true in one model if and only if it is true in the other. This means a specific pattern of knowledge and awareness is describable by one model exactly when it is describable by the other. The models start from different aspects of unawareness but end up expressing the same concept, which suggests they are capturing a natural idea of unawareness using minimal assumptions.

The relationships between models of unawareness are depicted in Figure 1. [5, Theorem 4.1] proved that the propositional model in [13] is equivalent to a special case (restricted to a single agent and propositionally generated awareness) of the logic of awareness in [4]. In addition, [6, Theorem 3.2] show the lattice of state spaces in [8] is equivalent to a special case (propositionally generated awareness) of the FH structure. The approach of [4] is proved equivalent to [14] by [18] and the quantified logic of awareness of [7] is proved equivalent to the [16] quantified impossible worlds model by [17].

According to [9] and [15], it is still an open question whether the Li structure is equivalent to those of [13] or [8]. This paper answers the question by showing that for any Li structure for modelling unawareness there is a FH structure (restricted to a single agent and propositionally generated awareness) expressing the same knowledge and awareness of the agent, and conversely for any propositionally generated single-agent FH structure there is a Li structure describing the same agent.

The equivalence proved in this paper allows comparison of the Li structure to additional models of unawareness.

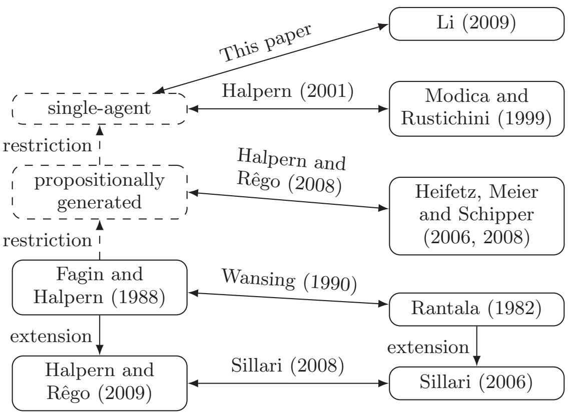

Figure 1: Relationships Between Models of Unawareness

Since [5] proved that the model of [13] is equivalent to a single-agent version of a propositionally generated logic of awareness and [6] proved that the multiagent model of [8] is equivalent to a propositionally generated FH structure, [11]'s model turns out to be equivalent to both [13] and a single-agent version of the model of [8, 9]. Since the models in the economics literature are proved equivalent to a special case of the FH structure, the approaches of [14, 4, 16, 7] in the artificial intelligence and philosophy literature are more general. Each of these, appropriately restricted, can describe the same knowledge and awareness as the models in economics.

The rest of this paper is organized as follows. In the next section the Li structure is presented in slightly more general form than in the original paper. To make it easier to show equivalence to the FH structure, the Li structure knowledge operator is modified and the new definition is proved equivalent to the initial one. In Section 3 the FH structure is presented and in Section 4 the main theorem is proved, showing the equivalence of the Li and the FH structures.

2. THE LI STRUCTURE

2.1 Primitives

The primitives of the model in [11] are a set of questions Q^* about relevant aspects of the world, a state space Ω^* consisting of vectors of answers to all the questions, an awareness function W^*, giving for each state the set of questions of which the agent is aware, and a possibility correspondence P^*, giving for each state the set of states that the agent thinks possible. Formally,

$$\Omega^* = \prod_{q \in Q^*} \{1_q, 0_q\}$$

$$W^* : \Omega^* \to 2^{Q^*}, \quad P^* : \Omega^* \to 2^{\Omega^*} \setminus \emptyset$$

Each state $\omega^* \in \Omega^*$ is a vector of zeros and ones with the interpretation that if a state contains a one in the coordinate corresponding to a question, the answer to that question is yes.

In the interests of later showing equivalence to the FH structure, the definition of the Li structure state space Ω^* is slightly generalized to allow some vectors of answers to be impossible and different states to have the same vector of answers. The states that have the same answers to all questions have labels to distinguish them that are drawn from some set S and are of the form s, s'. If it is necessary to refer to the labels, they are written as subscripts, e.g. ω^*_s. The new definition of the state space is

$$\Omega^* \subseteq \prod_{q \in Q^*} \{1_q, 0_q\} \times S$$

The definitions of the other primitives Q^*, W^* and P^* do not depend on the internal structure of Ω^* and are unaffected by the generalization of Ω^*.

Before defining the knowledge and awareness operators from the primitives introduced immediately above, some additional notation is needed.

Denote by $\omega^*(q)$ the q-th coordinate of ω^* and by $\omega^*(Q) = (\omega^*(q))_{q \in Q}$ the projection of ω^* onto the set of coordinates Q. For an event $E^* \subseteq \Omega^*$, denote by $E^*(Q)$ the set containing the projections of the states in E^* onto the coordinates in Q.

Subsequently Ω^* is called the objective state space and the states in it objective states. At each objective state ω^* the agent has a subjective state space $\Omega(\omega^*)$, which is the projection of the objective state space on the questions that the agent is aware of at ω^*. The subjective state space is defined as

$$\Omega(\omega^*) = \Omega^*(W^*(\omega^*))$$

The states in a subjective state space are denoted ω and called subjective states. Similarly to objective states, these can be projected onto a subset Q of their coordinates, with the result denoted $\omega(Q)$. The events E in a subjective state space are called subjective events and can be projected like objective events.

All the information needed in this paper about any objective or subjective event is (a) the objective states that project into it (the factual information) and (b) the questions determining the state space it belongs to (the awareness information). For any ω^* and $E \subseteq \Omega(\omega^*)$, denote the objective states projecting into E by E^* and the questions defining the space of E by $q(E)$. The formal definitions are

$$\begin{aligned}
&q : \cup_{\omega^* \in \Omega^*} 2^{\Omega(\omega^*)} \to 2^{Q^*} \\
&\Omega(\omega^*) \supseteq E \mapsto q(E) = W^*(\omega^*) \\
&E^* = \{\omega^* \in \Omega^* : \omega^*(q(E)) \in E\}
\end{aligned}$$

Any event can be written as a pair, $E = (E^*, q(E))$. If E is objective, then $E^* = E$ and $q(E) = Q^*$.

Example 1. *There is one question: "Is there a lamp in the cave?" so $Q^* = \{q_1\}$. There are two objective states: $\Omega^* = \{0, 1\}$ (there is no lamp or there is one). The possibility correspondence is $P^*(0) = \{0\}$, $P^*(1) = \{1\}$ (if the agent were fully aware, he would consider only the actual situation possible) and the awareness function is $W^*(0) = \emptyset$, $W^*(1) = \{q_1\}$ (if the cave has no lamp, the agent is unaware of its possibility, but if there is a lamp, then seeing it makes both the question and its answer obvious). The subjective state spaces are $\Omega(0) = \{x\}$ and $\Omega(1) = \{0, 1\}$.*

Consider four subjective events: $E_1 = \{x\}$, $E_2 = \{0, 1\}$, $E_3 = \{1\}$ and $E_4 = \{0\}$. The factual information of these events is $E_1^* = E_2^* = \{0, 1\}$, $E_3^* = \{1\}$ and $E_4^* = \{0\}$. The awareness information is $q(E_1) = \emptyset$, $q(E_2) = q(E_3) = q(E_4) = \{q_1\}$.

2.2 Awareness and subjective knowledge from the agent's perspective

Next, the unawareness operator and one of the subjective knowledge operators of the Li structure are introduced. Unawareness of an event E is defined as the states in which the agent is not aware of all the questions defining E. The agent cannot tell that each subjective state actually consists of multiple objective states and is not aware of all distinctions between the states in E and those in its complement E^c. The unawareness operator is

$$U(E) = \{\omega^* : q(E) \not\subseteq W^*(\omega^*)\}$$

Awareness is the complement of unawareness: it is the states in which all questions of E are in the agent's awareness set, formally

$$W(E) = \{\omega^* : q(E) \subseteq W^*(\omega^*)\} = \neg U(E)$$

The only information about an event that is relevant to the unawareness operator is the questions determining the state space of the event.

To provide background for the knowledge operators of [11] and to better compare them to the belief operators of [4], I first formalize an operator called objective knowledge. This is the traditional knowledge operator extended to subjective events.

$$K^*(E) = \{\omega^* : P^*(\omega^*) \subseteq E^*\}$$

The only information about an event that is relevant to the objective knowledge operator is the objective states projecting into the event.

Two knowledge operators are defined in [11]: subjective knowledge from the agent's perspective at state ω^*, denoted $\tilde{K}_{\omega^*}$, and subjective knowledge from the modeller's perspective, denoted K and defined via $\tilde{K}_{\omega^*}$. To define $\tilde{K}_{\omega^*}$, first a subjective version of the possibility correspondence P^* is needed.

Fix ω^* and let the projection of the possibility set at ω^* onto the subjective state space at ω^* be

$$R_{\omega^*} = \left\{\omega^{*\prime}(W^*(\omega^*)) : \omega^{*\prime} \in P^*(\omega^*)\right\}$$

For each ω^* a different set R_{ω^*} is defined. The subjective possibility correspondence at the objective state ω^* is defined in [11, Eq. (2.4)] as $P_{\omega^*}(\omega) = R_{\omega^*}$ if $\omega \in R_{\omega^*}$, and left undefined for $\omega \notin R_{\omega^*}$. There is a different function $P_{\omega^*}(\cdot)$ at each ω^*.

As shown below in Proposition 1, it is necessary to extend the definition by $P_{\omega^*}(\omega^*(W^*(\omega^*))) = R_{\omega^*}$ for a claim in [11] to hold. As Proposition 2 demonstrates, this extension is sufficient for the proofs of the results in the present paper.

The agent's subjective knowledge from his own perspective can now be defined using the subjective possibility correspondence. The subjective states at which an event E from some state space of the Li structure is known by an agent in ω^* are

$$\tilde{K}_{\omega^*}(E) = \begin{cases} \hat{K}_{\omega^*}(E) & \text{if } q(E) \subseteq W^*(\omega^*) \\ \emptyset_E & \text{if } q(E) \not\subseteq W^*(\omega^*) \end{cases} \quad (1)$$

$$\hat{K}_{\omega^*}(E) = \left\{\omega \in \Omega(\omega^*) : P_{\omega^*}(\omega) \subseteq \left\{\omega' \in \Omega(\omega^*) : \omega'(q(E)) \in E\right\}\right\}$$

where $\emptyset_E^* = \emptyset$ and $q(\emptyset_E) = q(E)$. Subjective knowledge from the agent's perspective can be iterated the standard way, $\tilde{K}^n_{\omega^*}(E) = \tilde{K}_{\omega^*}(\tilde{K}^{n-1}_{\omega^*}(E))$.

Example 2. *Continuing the previous example, the agent is aware of E_1 in both states, but aware of E_2, E_3, E_4 only in state 1. Due to the agent only considering the actual state possible, objective knowledge of each event equals the event, i.e. $K^*(E_i) = E_i$ in this model.*

To construct subjective knowledge from the agent's perspective, the subjective possibility correspondence is needed. Projecting the objective possibility correspondence, $R_0 = \{x\}$ and $R_1 = \{1\}$. From this, $P_1(0)$ is left undefined, $P_0(x) = \{x\}$ and $P_1(1) = \{1\}$. Due to the lack of awareness at state 0, $\tilde{K}_0(E_2) = \tilde{K}_0(E_3) = \tilde{K}_0(E_4) = \emptyset$. The only knowledge in state 0 is of event E_1: $\tilde{K}_0(E_1) = \{x\}$. At state 1, the undefined possibility correspondence at subjective state 0 means that $\tilde{K}_1(E_4) = \emptyset$, but the other events are known: $\tilde{K}_1(E_1) = \tilde{K}_1(E_2) = \tilde{K}_1(E_3) = \{1\}$.

2.3 Subjective knowledge from the modeller's perspective

There are two definitions of subjective knowledge from the modeller's perspective in [11]. The first defines it for all orders of iteration via the agent's subjective knowledge.

$$K_I^n(E) = \left\{\omega^* \in \Omega^* : \omega^*(W^*(\omega^*)) \in \tilde{K}^n_{\omega^*}(E)\right\} \quad (2)$$

Note that for each ω^* a corresponding $\tilde{K}_{\omega^*}$ is used in the condition $\omega^*(W^*(\omega^*)) \in \tilde{K}^n_{\omega^*}(E)$ to verify whether E is subjectively known at ω^*.

The second definition of subjective knowledge is given in [11] only for first-order knowledge.

$$K_{II}(E) = \{\omega^* : P^*(\omega^*) \subseteq E^*,\ q(E) \subseteq W^*(\omega^*)\} \quad (3)$$

The two definitions are stated in [11] to be equivalent for first order knowledge. In the present paper the equivalence of the definitions is proved for all orders of knowledge. The second definition for higher order knowledge is

$$K_{II}^n(E) = \left\{\omega^* : P^*(\omega^*) \subseteq K_{II}^{n-1}(E),\ q(E) \subseteq W^*(\omega^*)\right\} \quad (4)$$

Note that $q(E)$, not $q(K^{n-1}(E))$ is used in the definition. Since the event $K(E)$ is objective by both (2) and (4), it contains maximal awareness information. Thus iterating knowledge $K^n(E) = K(K^{n-1}(E))$ would require full awareness for any higher order knowledge.

Example 3. *Continuing the example, subjective knowledge from the modeller's perspective as the conjunction of awareness and objective knowledge gives $K(E_1) = \{0, 1\}$, $K(E_2) = K(E_3) = \{1\}$ and $K(E_4) = \emptyset$. Li's definition of subjective knowledge from the modeller's perspective (objectivizing $\tilde{K}$) gives the same result.*

In this example, higher order subjective knowledge is the same as first order: $KK(E_i) = K(E_i)$.

In the following proposition I show that the two definitions (2) and (4) are equivalent for first-order knowledge, as claimed in [11]. The proof uses the extension of the definition of the subjective possibility correspondence given above.

Proposition 1. *The operators K_I and K_{II} coincide in all Li structures iff $P_{\omega^*}(\omega^*(W^*(\omega^*))) = R_{\omega^*}$.*

Proof. The definitions K_I and K_{II} coincide iff

$$\omega^*(W^*(\omega^*)) \in \tilde{K}_{\omega^*}(E) \Leftrightarrow \quad (5)$$
$$P^*(\omega^*) \subseteq E^* \wedge q(E) \subseteq W^*(\omega^*)$$

By definition (1) of $\tilde{K}_{\omega^*}$, the condition $q(E) \subseteq W^*(\omega^*)$ appears on both sides of (5), which reduces to

$$P_{\omega^*}(\omega) \subseteq \left\{\omega' \in \Omega(\omega^*) : \omega'(q(E)) \in E\right\} \Leftrightarrow P^*(\omega^*) \subseteq E^* \quad (6)$$

where ω is the subjective state in the space $\Omega(\omega^*)$ that ω^* projects to, i.e. $\omega = \omega^*(W^*(\omega^*))$.

If the agent is deluded, i.e. $\omega^* \notin P^*(\omega^*)$, and the condition $P_{\omega^*}(\omega^*(W^*(\omega^*))) = R_{\omega^*}$ is not satisfied, then the subjective state $\omega^*(W^*(\omega^*))$ is outside the scope of the original definition of the subjective possibility correspondence. This leaves K_{II} undefined and therefore K_I and K_{II} do not coincide in Li structures with a delusional agent. Necessity of the extension $P_{\omega^*}(\omega^*(W^*(\omega^*))) = R_{\omega^*}$ is thus proven.

For sufficiency, note that $\{\omega' \in \Omega(\omega^*) : \omega'(q(E)) \in E\}$ is the projection of E^* into $\Omega(\omega^*)$ and if the condition in the statement of the proposition holds, then $P_{\omega^*}(\omega^*(W^*(\omega^*)))$ is the projection of $P^*(\omega^*)$ into $\Omega(\omega^*)$. Thus any definition of the subjective possibility correspondence that makes R_{ω^*} the possibility set at $\omega^*(W^*(\omega^*))$ will ensure that (6) holds, since subset relations are preserved by projections. □

The following proposition proves that K_{II}^n as defined in (4) is equivalent to the definition (2) under the condition given in Proposition 1.

Proposition 2. *If $P_{\omega^*}(\omega^*(W^*(\omega^*))) = R_{\omega^*}$, then the operators K_I^n and K_{II}^n are equivalent.*

The proof, which is relegated to the appendix, reduces both expressions to the same function of objective knowledge and awareness

$$K^{*n}(E) \cap K^{*n-1}W(E) \cap ... \cap K^*W(E) \cap W(E)$$

Note that this expression reduces to iterated objective knowledge $K^{*n}(E)$ if the agent is fully aware in every state, i.e. in the standard model with one state space. This is to be expected, as the subjective knowledge operators are meant to model an agent who has limited awareness. If such an agent becomes fully aware, his knowledge operator should become the standard one in models without unawareness.

In the rest of the paper the definition K_{II} is used and the operator is denoted K.

3. THE FH STRUCTURE

Fagin and Halpern describe two intertwined models with the same content—a set-based model called the Kripke structure for awareness and a propositional model called the logic of awareness. This paper restricts the FH structure to a single agent and to propositionally generated awareness, which will be defined below.

The primitives of the FH structure are a set of propositions Φ, logical operators $\neg$ for negation and $\wedge$ for conjunction, modal operators A for awareness, L for implicit belief and B for explicit belief, a state space S, an awareness function $\mathcal{A}$ giving for each state the propositions of which the agent is aware, a possibility correspondence $\mathcal{B}$ giving for each state the states the agent considers possible, and a truth assignment π that for each state-proposition pair (s, p) takes the value 1 if proposition p is true in state s and 0 otherwise. Formally,

$$\mathcal{A} : S \to 2^{\Phi}, \quad \mathcal{B} : S \to 2^{S} \setminus \emptyset, \quad \pi : S \times \Phi \to \{0, 1\}$$

The logical and modal operators are combined with primitive propositions in an inductive way to construct well-formed formulas (wff) denoted by ϕ, ψ. The rules to construct well-formed formulas are

$$\begin{array}{l} \text{All propositions } p \in \Phi \text{ are wff} \\ \text{If } \phi \text{ and } \psi \text{ are wff, then } \neg\phi \text{ and } \phi \wedge \psi \text{ are wff} \\ \text{If } \phi \text{ is a wff, then } L\phi, B\phi \text{ and } A\phi \text{ are wff} \end{array} \qquad (7)$$

An example of a wff is $\phi = \neg BL(\neg p)$, interpreted as 'the agent does not explicitly believe that he implicitly believes that p is false'. For any formula ϕ, denote the set of propositions found in ϕ by $\mathrm{Prim}(\phi)$.

The Kripke structure for awareness is the set-based part of the FH structure. It consists of S, $\mathcal{A}$, $\mathcal{B}$ and π, which were defined above. The logic of awareness and the Kripke structure are connected by assigning truth values to all formulas of the logic in every state. Denote a formula ϕ being true in a state s of a Kripke structure M by $(M, s) \models_{FH} \phi$. The truth assignment rules are as follows.

(a) $(M, s) \models_{FH} p$ iff $\pi(s, p) = 1$

(b) $(M, s) \models_{FH} \neg\phi$ iff $(M, s) \not\models_{FH} \phi$

(c) $(M, s) \models_{FH} \phi \wedge \psi$ iff $(M, s) \models_{FH} \phi$ and $(M, s) \models_{FH} \psi$

(d) $(M, s) \models_{FH} A\phi$ iff $\mathrm{Prim}(\phi) \subseteq \mathcal{A}(s)$

(e) $(M, s) \models_{FH} L\phi$ iff $\forall t \in \mathcal{B}(s), \quad (M, t) \models_{FH} \phi$

(f) $(M, s) \models_{FH} B\phi$ iff $\mathrm{Prim}(\phi) \subseteq \mathcal{A}(s)$ and $\forall t \in \mathcal{B}(s)$, $(M, t) \models_{FH} \phi$

The restriction on the FH structure that awareness is propositionally generated is precisely the truth assignment rule to formulas of the form $A\phi$. By [7], this rule is equivalent to the agent being aware of a formula ϕ iff he is aware of all propositions in ϕ, written as $A\phi = \wedge_{p \in \mathrm{Prim}(\phi)} Ap$. An interpretation of propositionally generated awareness is that the agent's language is limited and cannot describe all aspects of the world.

The agent's awareness and knowledge in the FH structure are described by the truth values of formulas containing the awareness and explicit belief operators. The agent is said to explicitly believe the formula ϕ in state s if the formula $B\phi$ is true in state s. The agent is aware of ψ in s if $A\psi$ is true in s. For each formula ϕ the set of states where it is true is called its truth set and denoted $\|\phi\|$, with the formal definition $\|\phi\| = \{s : (M, s) \models_{FH} \phi\}$.

4. EQUIVALENCE OF THE STRUCTURES

In this section it is proved that the information structure with unawareness of [11] is equivalent in a strong sense to the special case of the logic of awareness of [4] where awareness is propositionally generated and there is a single agent. For every Li structure one can construct a FH structure describing the same knowledge and awareness of the agent, and conversely for every FH structure there is a Li structure that describes the same knowledge and awareness.

To make precise what is meant by 'describing the same knowledge and awareness', a logic is attached to the Li structure. Two structures express the same knowledge and awareness if the same formulas of the same logic are true in the corresponding states in both structures.

The logic attached to the Li structure allows expression of exactly the same formulas as the logic of awareness, but the truth assignment to the formulas uses different rules. The primitives of the logic are a set of propositions Φ with the same number of propositions as there are questions in Q^*, logical operators $\neg$ and $\wedge$, and modal operators A, L and B. Well-formed formulas are constructed by the standard rules (7).

To connect the logic to the Li structure via a truth assignment, first a correspondence (denoted $\leftrightarrow$) is inductively defined between the formulas of the logic and the subjective events of the Li structure. The correspondence is based on the fact that both formulas and subjective events carry two types of relevant information. A formula has a truth set and the primitive propositions contained in the formula. A subjective event consists of the objective states contained in it and the questions defining its state space. Table 1 gives an overview of the correspondence.

As a starting point for the correspondence, the propositions in Φ are indexed by the same set as the questions in Q^*, so for each proposition p_i there is a unique corresponding question q_i. A natural equivalence between propositions and questions can be created if each proposition p_i is interpreted 'q_i is answered affirmatively' and each q_i is interpreted as 'Is p_i true?'

The base case of the induction is defining for each proposition p_i its truth set as $\|p_i\| = \{\omega^* : \omega^*(q_i) = 1\}$ and the subjective event corresponding to p_i as $E = (\|p_i\|, \{q_i\}) \leftrightarrow p_i$, so that the objective states in E are the truth set of p_i and the only question q_i of E has the same index as p_i. The inductive steps extend the proposition-event correspondence to all formulas and all subjective events.

Denote the complement of a set G as G^c. Assuming $\phi \leftrightarrow (E^*, q(E))$ and $\psi \leftrightarrow (F^*, q(F))$, the induction on the length of formulas is

(a) $\neg\phi \leftrightarrow (E^{*c}, q(E))$

(b) $\phi \wedge \psi \leftrightarrow (E^* \cap F^*, q(E) \cup q(F))$

(c) $A\phi \leftrightarrow (W(E), q(E))$

(d) $L\phi \leftrightarrow (K^*(E), q(E))$

(e) $B\phi \leftrightarrow (K(E), q(E))$

By the definition of $\leftrightarrow$ it is clear that if $E \leftrightarrow \phi$, then $\{i : q_i \in q(E)\} = \{j : p_j \in \mathrm{Prim}(\phi)\}$, which is denoted as $q(E) = \mathrm{Prim}(\phi)$ with some abuse of notation. This fact is used in the proof of the main theorem to establish the equivalence of awareness operators in the Li and FH structures.

Truth assignment in the Li structure uses the set operators W, K and K^* defined in Section 2 and the correspondence $\leftrightarrow$ between formulas and subjective events defined above. The notation $(M, \omega^*) \models_{Li} \phi$ means formula ϕ is true in objective state ω^* in Li structure M.

(a) $(M, \omega^*) \models_{Li} p$ iff $\omega^* \in \|p\|$

(b) $(M, \omega^*) \models_{Li} \neg\phi$ iff $\omega^* \in \|\phi\|^c$

(c) $(M, \omega^*) \models_{Li} \phi \wedge \psi$ iff $\omega^* \in \|\phi\| \cap \|\psi\|$

Table 1: Correspondence of subjective events and formulas

Primitives		Operators		Subjective events and formulas	
Li	FH	Li	FH	Li	FH
Q^*	Φ	$(E^{*c}, q(E))$	$\neg\phi$	$q(E)$	$\mathrm{Prim}(\phi)$
Ω^*	S	$(E^* \cap F^*, q(E) \cup q(F))$	$\phi \wedge \psi$	E^*	$\|\phi\|$
$\omega^*(q)$	$\pi(s,p)$	$(W(E), q(E))$	$A\phi$	E	ϕ
W^*	$\mathcal{A}$	$(K^*(E), q(E))$	$L\phi$		
P^*	$\mathcal{B}$	$(K(E), q(E))$	$B\phi$		

(d) $(M, \omega^*) \models_{Li} A\phi$ iff $\omega^* \in W(E)$ and $E \leftrightarrow \phi$

(e) $(M, \omega^*) \models_{Li} L\phi$ iff $\omega^* \in K^*(\|\phi\|)$

(f) $(M, \omega^*) \models_{Li} B\phi$ iff $\omega^* \in K(E)$ and $E \leftrightarrow \phi$

These truth assignment rules are determined by the set-based operators in the Li structure to ensure that the logic describes exactly the same knowledge and awareness as the events in the structure.

The following theorem establishes the main result of the paper, the equivalence between the Li structures and the appropriately restricted FH structures.

THEOREM 3. *(a) For every Li structure $M = (Q^*, \Omega^*, P^*, W^*)$ there exists a FH structure $M' = (\Phi, S, \mathcal{B}, \mathcal{A}, \pi)$ such that for all formulas ϕ constructed from Φ and $\neg$, $\wedge$, L, B, A using the rules (7), it is the case that $(M, \omega^*) \models_{Li} \phi$ iff $(M', s) \models_{FH} \phi$.*

(b) For every FH structure $M' = (\Phi, S, \mathcal{B}, \mathcal{A}, \pi)$ there is a Li structure $M = (Q^*, \Omega^*, P^*, W^*)$ such that for all formulas ϕ constructed from Φ and $\neg$, $\wedge$, L, B, A using the rules (7) we have $(M', s) \models_{FH} \phi$ iff $(M, \omega^*) \models_{Li} \phi$.

The proof proceeds by induction on the complexity of formulas. First the primitives of the two models are set to correspond in a natural way. Then truth values are assigned to formulas by the rules of the two models. The correspondence of the primitives directly implies that the truth values of propositions are the same in the Li and FH structures, which is the base case of the induction. Finally it is shown that the truth assignments agree on all formulas. This is done by induction on the length of formulas: assuming the truth values of formulas of a given length coincide in the Li and FH structures, it is shown that the truth values of the longer formulas obtained by applying one of the operators $\neg$, $\wedge$, A, B or L are the same.

PROOF. For part (a), take a set Φ with the same number of propositions as questions in Q^* and index the propositions so that p_i corresponds to q_i. Define the truth assignment to propositions π as $\pi(\omega^*, p) = \omega^*(p)$. Construct the set of formulas from Φ by the rules (7). Take $S = \Omega^*$, $\mathcal{B} = P^*$ and $\mathcal{A}$ such that $\{i : p_i \in \mathcal{A}(\omega^*)\} = \{j : q_j \in W^*(\omega^*)\}$. Abusing notation, this can be written $\mathcal{A} = W^*$. Assign truth values to formulas in the Li structure and the FH structure by their respective truth assignment rules.

Proceed by induction on the length of formulas to show that the truth assignment rules in the two structures are equivalent, which establishes the conclusion that each formula is true in the Li structure iff it is true in the FH structure.

The base case of the induction is truth assignment to the primitive propositions. By the first truth assignment rule in the Li structure, $(M, \omega^*) \models_{Li} p$ iff $\omega^* \in \|p\| = \{\omega^* : \omega^*(p) = 1\}$, by construction $\pi(\omega^*, p) = \omega^*(p)$, and by the first rule in the FH structure, $(M, s) \models_{FH} p$ iff $\pi(s, p) = 1$. Putting the three equivalences together results in $(M, \omega^*) \models_{Li} p$ iff $(M, s) \models_{FH} p$.

For negations, equivalence of the truth assignment rules follows from $(M, \omega^*) \models_{Li} \neg\phi$ iff $\omega^* \in \|\phi\|^c$ and $(M, s) \models_{FH} \neg\phi$ iff $(M, s) \not\models_{FH} \phi$. Under both truth assignment rules, $\neg\phi$ is true exactly when ϕ is false.

A similar reasoning establishes the equivalence for conjunctions. $(M, \omega^*) \models_{Li} \phi \wedge \psi$ iff $\omega^* \in \|\phi\| \cap \|\psi\|$, which holds iff $\omega^* \in \|\phi\|$ and $\omega^* \in \|\psi\|$, exactly when $(M, \omega^*) \models_{Li} \phi$ and $(M, \omega^*) \models_{Li} \psi$. Assuming the result to be proved holds for all formulas shorter than $\phi \wedge \psi$ and using the rule $(M, s) \models_{FH} \phi \wedge \psi$ iff $(M, s) \models_{FH} \phi$ and $(M, s) \models_{FH} \psi$, it is clear that $(M, s) \models_{FH} \phi \wedge \psi$ iff $(M, s) \models_{Li} \phi \wedge \psi$.

For formulas expressing awareness, the truth assignment rules are $(M, \omega^*) \models_{Li} A\phi$ iff $\omega^* \in W(E)$, $E \leftrightarrow \phi$ in the Li structure and $(M, s) \models_{FH} A\phi$ iff $\mathrm{Prim}(\phi) \subseteq \mathcal{A}(s)$ in the FH structure. The set-based awareness operator W was defined in Section 2 as $W(E) = \{\omega^* : q(E) \subseteq W^*(\omega^*)\}$. By construction, $W^*(\omega^*) = \mathcal{A}(\omega^*)\ \forall \omega^*$, and by definition of $\leftrightarrow$, $q(E) = \mathrm{Prim}(\phi)$ for $E \leftrightarrow \phi$, so for any state ω^* it is the case that $q(E) \subseteq W^*(\omega^*) \Leftrightarrow \mathrm{Prim}(\phi) \subseteq \mathcal{A}(\omega^*)$. From this it follows that $(M, \omega^*) \models_{Li} A\phi$ iff $(M, s) \models_{FH} A\phi$.

The truth assignment to implicit belief in the Li structure is $(M, \omega^*) \models_{Li} L\phi$ iff $\omega^* \in K^*(\|\phi\|)$. According to $K^*(E) = \{\omega^* : P^*(\omega^*) \subseteq E^*\}$, the definition of objective knowledge, the condition $\omega^* \in K^*(\|\phi\|)$ can be equivalently expressed as $P^*(\omega^*) \subseteq \|\phi\|$. The possibility correspondences in the two structures are equal by construction, so the condition becomes $\mathcal{B}(\omega^*) \subseteq \|\phi\|$, i.e. ϕ is true in all states of $\mathcal{B}(\omega^*)$. This is the condition for the truth assignment to implicit belief formulas in the FH structure, $(M, s) \models_{FH} L\phi$ iff $\forall t \in \mathcal{B}(s), \quad (M, t) \models \phi$.

For formulas expressing explicit belief, the Li structure truth assignment requires $\omega^* \in K(E)$ and $E \leftrightarrow \phi$. Since $K(E) = W(E) \cap K^*(E)$, an explicit belief formula $B\phi$ is true in a state of the Li structure iff the formulas expressing awareness and implicit belief of the same statement, $A\phi$ and $L\phi$ are true in that state. Similarly in the FH structure the truth condition $\mathrm{Prim}(\phi) \subseteq \mathcal{A}(s)$ and $\forall t \in \mathcal{B}(s), \quad (M, t) \models \phi$ for an explicit belief formula is a conjunction of the conditions for the awareness and implicit belief formulas for the same concept. The equivalence of the truth assignments to explicit belief formulas in the two structures is a direct consequence of the equivalence for the formulas expressing awareness and implicit belief.

For part (b), take a set Q^* with the same number of questions as propositions in Φ and index the questions so that q_i corresponds to p_i. If no two states in S have the same truth assignment to propositions, the states in the Li structure could be defined as the truth assignments to states in the FH structure. If truth assignments repeat for states in S, then vectors of answers to questions in Q^* will repeat in Ω^*. It is possible to distinguish states with the same answer profile by labeling each $\omega^* \in \Omega^*$ with the $s \in S$ that it is meant to correspond to. Therefore to construct a Li structure equivalent to a FH structure that may have repeating truth assignments, define

$$\Omega^* = \{((\pi(s,p_i))_{p_i \in \Phi}, s) : s \in S\} =$$
$$\{\omega_s^* \in \prod_{q \in Q^*} \{1_q, 0_q\} \times S : \exists s \in S,\ \forall i\ \omega_s^*(q_i) = \pi(s,p_i)\}$$

The last coordinate of each state in Ω^* is written as a subscript, e.g. ω_s^*, for easier subsequent reference.

The possibility correspondence and awareness function in the Li structure are constructed to contain the same information as the possibility and awareness operators in the FH structure. The possibility correspondence is defined as $P^*(\omega_s^*) = \{\omega_{s'}^* : s' \in \mathcal{B}(s)\}$ and the awareness function $W^*(\omega_s^*) = \{q_i \in Q^* : p_i \in \mathcal{A}(s)\}$, where s ranges over S and therefore ω_s^* ranges over Ω^*. Abusing notation, write $P^* = \mathcal{B}$ and $W^* = \mathcal{A}$.

The induction on the length of formulas in part (b) to show that the truth assignments in the two structures are equivalent is exactly the same as in the proof of part (a), since each step of the induction in part (a) showed implication both ways. □

According to Theorem 3, Li structures provide an alternative notation for the single-agent propositionally generated logic of awareness and its Kripke structure described in Section 3. The state space, possibility correspondence and awareness function in a Li structure are equivalent to a Kripke structure for awareness, so these can be used to provide an alternative semantics for the logic of awareness (as is done in [5] with the model of [13]).

The propositionally generated FH structure can describe a multiagent setting without further modification besides the addition of possibility correspondences and awareness functions for more agents. Since the Li structure is equivalent to a FH structure, adding the same elements to the Li structure will extend it to the multiagent case. The equivalence proof thus provides an avenue to extend the construction of [11] based on [4].

In the other direction it would be interesting to find the object in the FH structure that corresponds to subjective knowledge from the agent's perspective. The construction in (1) or the agent's view in general seems to have no natural counterpart in logic or semantics.

5. CONCLUSION

I have shown that the set-based model of unawareness of [11] is equivalent (in the usual sense of equivalence in the literature) to a single-agent version of the propositionally generated logic of awareness presented by [4]. For any model in one of these categories it is possible to construct a model in the other category describing the same agent—we can translate from one model to the other. Therefore any notion of knowledge or awareness expressible in the Li structure is expressible in the FH structure and vice versa.

The close connection between the propositional model of [4] and the set-based model of [11] is interesting for a technical and a conceptual reason. From a technical point of view, the equivalence demonstrated in the present paper highlights the features of knowledge and awareness expressible in both set-based and propositional notation, and those describable in only the more general propositional models.

From a conceptual perspective, the Li and FH structures start from different aspects of unawareness (the inability to distinguish states and the limited language) and end up describing the same notion. This suggests a linkage between different properties of unawareness, so that any one of them implies all the others. [11] starts by separating the subjective and objective state spaces, coarsening the agent's view of the world relative to an objective description of the environment. [11]'s unawareness operator captures an inability to distinguish states whose descriptions differ only on dimensions the agent is not able to reason about. [4] limit the agent's language, permitting reasoning only in terms of propositions available to the agent. Both approaches separate the analyst's and the agent's perspectives (the objective and subjective views of the world). Both models result in a nontrivial unawareness operator that satisfies the properties suggested by [3], thus bypassing the impossibility result.

Through its equivalence with the FH structure, the Li structure is also equivalent (in the usual sense) to the model of [13] and a single-agent version of [8], which similarly model nontrivial unawareness. The proof of equivalence in the present paper answers the question posed in [9] and [15] about the possible connection between the four models of unawareness.

The equivalence between the inability to perceive some aspects of the environment and the inability to reason about some propositions vindicates Wittgenstein's claim 'The limits of my language mean the limits of my world'. One open question is which other limitations of language and perception this equivalence is obtained for—Wittgenstein's claim need not always hold. A higher level equivalence between distinguishability and expressibility may shed light on the connection between the mainly physical notion of being able to sense and measure parameters of the environment and the cognitive notion of being able to think about and express these parameters.

Since the logic of awareness has been augmented with quantification in [7] to describe awareness of unawareness, a possible direction for further study is to extend Li's set-based model to express this as well, if possible. The potential equivalence of the extended Li structure with the quantified logic of awareness can then be examined. This may give an idea about the extent of the equi-expressivity between propositional and set-based models.

Awareness of unawareness is one form of uncertainty between ambiguity and unawareness. A natural question is whether there are other types of uncertainty with less higher order knowledge than ambiguity and more than unawareness. The agent may have an idea about the sets of objects where he is unaware of some elements, e.g. know there exist both viruses and bacteria he is unaware of, with some properties of both domains known. To compare the various types of uncertainty it may be useful to embed them in a general infinite hierarchy of uncertainty, extending the hierarchies for probability [12], conditional probability [2] and ambiguity [1].

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APPENDIX

A. PROOF OF PROPOSITION 2

Proposition 2 states that if $P_{\omega^*}(\omega^*(W^*(\omega^*))) = R_{\omega^*}$, then the operators K_I^n and K_{II}^n are equivalent.

In the proof it will be shown that $K_I^n(E)$ and $K_{II}^n(E)$ both equal to the same function of awareness and objective knowledge

$$K^{*n}(E) \cap K^{*n-1}W(E) \cap ... \cap K^*W(E) \cap W(E)$$

The reduction for $K_{II}^n(E)$ uses the conjunction property for objective knowledge

$$K^*(F \cap G) = K^*(F) \cap K^*(G)$$

which is immediate from the definition of K^*.

$$\begin{aligned}
&K_{II}^n(E) = \left\{\omega^* : P^*(\omega^*) \subseteq K_{II}^{n-1}(E)\right\} \cap W(E) = \\
&K^*(K_{II}^{n-1}(E)) \cap A(E) = \\
&K^*\left(\left\{\omega^* : P^*(\omega^*) \subseteq K_{II}^{n-2}(E)\right\} \cap W(E)\right) \cap W(E) = \\
&K^{*2}\left(K_{II}^{n-2}(E)\right) \cap K^*W(E) \cap W(E) = ... = \\
&K^{*n}(E) \cap K^{*n-1}W(E) \cap ... \cap K^*W(E) \cap W(E)
\end{aligned}$$

For K_I^n, use definition (1) of $\tilde{K}_{\omega^*}$ to rewrite it as

$$K_I^n(E) = \left\{\omega^* : \omega^*(W^*(\omega^*)) \in \tilde{K}_{\omega^*}(\tilde{K}_{\omega^*}^{n-1}(E))\right\} =$$

$$\begin{cases}
\left\{\omega^* : P_{\omega^*}(\omega^*(W^*(\omega^*))) \subseteq \tilde{K}_{\omega^*}^{n-1}(E)\right\} \\
\quad \text{if } q(\tilde{K}_{\omega^*}^{n-1}(E)) \subseteq W^*(\omega^*) \\
\left\{\omega^* : \omega^*(W^*(\omega^*)) \in \emptyset_{\tilde{K}_{\omega^*}^{n-1}(E)}\right\} \\
\quad \text{if } q(\tilde{K}_{\omega^*}^{n-1}(E)) \not\subseteq W^*(\omega^*)
\end{cases}$$

Next, the awareness condition is simplified by noting that, based on the definition of $\tilde{K}_{\omega^*}$,

$$q(\tilde{K}_{\omega^*}(E)) = \begin{cases} W^*(\omega^*) & \text{if } q(E) \subseteq W^*(\omega^*) \\ q(E) & \text{if } q(E) \not\subseteq W^*(\omega^*) \end{cases}$$

from which it follows that $q(\tilde{K}_{\omega^*}(E)) \subseteq W^*(\omega^*)$ if and only if $q(E) \subseteq W^*(\omega^*)$. Repeating this argument leads to $q(\tilde{K}_{\omega^*}^{n-1}(E)) \subseteq W^*(\omega^*)$ if and only if $q(E) \subseteq W^*(\omega^*)$. This simplifies K_I^n to

$$K_I^n(E) = \begin{cases}
\left\{\omega^* : P_{\omega^*}(\omega^*(W^*(\omega^*))) \subseteq \tilde{K}_{\omega^*}^{n-1}(E)\right\} \\
\quad \text{if } q(E) \subseteq W^*(\omega^*) \\
\emptyset \quad \text{if } q(E) \not\subseteq W^*(\omega^*)
\end{cases}$$

Due to the preservation of subset relations by projections, $P_{\omega^*}(\omega^*(W^*(\omega^*))) \subseteq \tilde{K}^{n-1}_{\omega^*}(E)$ reduces to $P^*(\omega^*) \subseteq (\tilde{K}^{n-1}_{\omega^*}(E))^*$ and

$K^n_I(E) = \Big\{\omega^* : P^*(\omega^*) \subseteq (\tilde{K}^{n-1}_{\omega^*}(E))^*,\ q(E) \subseteq W^*(\omega^*)\Big\} =$

$K^*\left((\tilde{K}^{n-1}_{\omega^*}(E))^*\right) \cap \{\omega^* : q(E) \subseteq W^*(\omega^*)\} =$

$K^*(K^{n-1}_I(E)) \cap W(E)$

Repeating the previous steps, the order of subjective knowledge in the expression can be again reduced by one.

$$K^n_I(E) = K^*\Bigg(\Big\{\omega^* : P^*(\omega^*) \subseteq (\tilde{K}^{n-2}_{\omega^*}(E))^*\Big\} \cap \{\omega^* : q(E) \subseteq W^*(\omega^*)\}\Bigg) \cap W(E)$$

Iterating this procedure, the end result will be $K^{*n}(E) \cap K^{*n-1}W(E) \cap ... \cap K^*W(E) \cap W(E)$ as desired. Since the two operators are both equal to the same expression, they are equivalent.