Syntactic Foundations for Unawareness of Theorems
- Extended Abstract -

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Abstract

We provide a complete and sound axiomatization of the set-theoretic model of Galanis [2007]. By constructing a syntax with several knowledge modalities, one for each sub-language, we are able to allow for agents to make mistakes about the knowledge of others without discarding the truth axiom. Comparing the present axiom system with that of Heifetz et al. [2008a] we find that neither is a generalization of the other.

JEL-Classifications: C70, C72, D80, D82.

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1 Introduction

We provide a complete and sound axiomatization of the set-theoretic model of Galanis [2007]. The approach we use follows that of Heifetz et al. [2008a] (HMS from now on), of constructing a canonical unawareness structure. The purpose of the axiomatization is to provide syntactic foundations for the set-theoretic model and to compare this approach with the other papers in the literature.

In order to illustrate the difference between the present and other approaches, we need to distinguish between a language and a sub-language. When modeling knowledge using a syntactic approach, the modeler starts with a set of primitive propositions, consisting of statements like "it rains" or "the price is high". Using negation ($\neg$), conjunction ($\wedge$) and the knowledge modality k^i, a language is created, containing all the well formed formulas. Moreover, it is implicitly assumed that all agents have a perfect understanding of that language. For example, the formula "agent i knows that it rains" is equally understood by everyone. However, if we introduce unawareness, this may not be true.

Modica and Rustichini [1999] and HMS specify that apart from the universal language that is generated from all primitive propositions, there are also several sub-languages, each generated by *some* of the primitive propositions. An agent who is aware only of some primitive propositions describes the world using one sub-language, which may be very different from the sub-language used by another agent. Moreover, the agents may not comprehend or be unaware of other sub-languages.

Suppose there are two agents, each using a different sub-language, both containing the statement "the price is high". It is clear that both agents understand "prices" in the same way. However, does this imply that the statement "agent i knows that the price is high" is also understood in the same way by both agents? In other words, is knowledge when described in one sub-language identical to knowledge when described in another sub-language? In HMS and other papers in the literature the answer is "yes", so there is only

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one, objective, knowledge modality. In Galanis [2007] and in this paper we allow for the knowledge modality to be different across sub-languages. This captures the idea that agents of different perception (awareness) may reason differently about the knowledge of others.

A short example illustrates the point.[1] Suppose that agent j is only aware of the statement "the price is high" and that the price will be determined tomorrow. In other words, it is simply not possible for anyone to know whether prices are high or low, if he is only aware of prices. Hence, agent j concludes that, in his sub-language, i does not know whether the price is high. But suppose that agent i is also aware of the statement "the interest rate is low" and that there is a logical connection specifying that a low interest rate implies a high price. Agent i concludes that, in his sub-language, he knows that the price is high. Although agent j is wrong when reasoning about i's knowledge, this is only because he is constrained by his unawareness. Because it is correct that no one can know whether the price is high if he is unaware of the interest rate, j is not making a mistake within the bounds of his awareness.

One way of modeling this example, within the standard setting of a unique knowledge modality, is to drop the truth axiom, ($k^j\phi \rightarrow \phi$), which says that if j knows something then it is true. However, in this way we allow agents to be totally irrational and to make all kinds of mistakes, even unrelated to unawareness.

In order to avoid this extra irrationality we introduce one knowledge modality, k^i_α, for each sub-language which is generated by a set α of primitive propositions. Moreover, we impose the truth axiom for each knowledge modality ($k^i_\alpha\phi \rightarrow \phi$) and we add an axiom saying that more complete sub-languages give a better description of knowledge. Formally, if $\alpha \subseteq \alpha'$ then $k^i_\alpha\phi \rightarrow k^i_{\alpha'}\phi$. Therefore, agent j can make a mistake about i's knowledge only if his sub-language is not more complete than i's sub-language. For example, we can simultaneously have $\neg k^i_\alpha\phi$ and $k^i_{\alpha'}\phi$ only if α' is not a subset of α.[2] Moreover, since the truth axiom holds for every sub-language, this is the only mistake in reasoning that any agent is allowed to make.

For instance, suppose j knows ϕ, so that $k^j_\alpha\phi$ is true. If ϕ is the statement "it rains" then it is true that it rains. But if ϕ is the statement $\neg k^i_\alpha\phi'$, then although $\neg k^i_\alpha\phi'$ is also true, it may be that because agent i's sub-language is α' (and $\alpha \subset \alpha'$) we also have $k^i_{\alpha'}\phi'$, so that j is (essentially) making a mistake.

Summarizing, although one could argue that an "awareness leads to knowledge" effect is better captured in a dynamic environment, it has significant implications in a static model as well. As was described above, if two agents differ only in that one is more aware than the other, then the more aware agent would have more knowledge. Moreover, the less aware agent would mistakenly think that their knowledge is the same. This can be captured by relaxing the truth axiom in the standard model but then all possible mistakes (even unrelated to unawareness) are allowed. The other possibility is introducing multiple knowledge modalities, as described above.

A few clarifications are in order. First, since there are many knowledge modalities, which is the one that provides the true description of the agent's knowledge? This depends on the agent's sub-language, which is determined by his awareness. Consequently, when agent i reasons about j's knowledge, he first has to reason about j's awareness and sub-language.

Second, more complete sub-languages give a better description of one's knowledge. The reason is that more complete sub-languages contain more knowledge modalities, $k^i_\alpha\phi$, $k^i_{\alpha'}\phi$, $k^i_{\alpha''}\phi$, describing i's knowledge about formula ϕ. Having multiple knowledge operators allows for the possibility that although a state specifies that an agent knows an event, the projection of that state to a less complete state space specifies that he does not know it. This is because the projection contains fewer knowledge operators and hence pro-

[1] In Galanis [2007] we provide more examples and arguments.

[2] In the example, i's sub-language is generated by primitive propositions in α' and j's sub-language is generated by α.

vides a less complete description of one's knowledge. Hence, knowledge can change across projections not because formulas in the projected states are allowed to differ, but because some formulas describing knowledge are not included in the projection. We elaborate on this point when we construct the canonical unawareness structure.

Third, we do not allow for false certainties. In other words, it is never the case that an agent knows a formula which is false. This is due to the truth axiom. At the same time, we allow agents to make statements which, from another agent's point of view with a richer awareness, are mistaken. We say that agent i makes a mistake in his reasoning about j if, for example, he is aware only of primitive propositions in α and knows that j is aware of α, he knows that $\neg k_{\alpha}^{j}\phi$ and yet it is true that $k_{\alpha'}^{j}\phi$ and agent j is aware of all propositions in α', where $\alpha \subset \alpha'$. Because $\neg k_{\alpha}^{j}\phi$ is also true, the truth axiom is not violated. Moreover, i is not making a mistake when reasoning that j's awareness is α, when in fact it is α'. The reason is that i is only aware of primitive propositions in α, so he cannot reason above that level.

Comparing the present axiom system with that of HMS, we find two main differences. First, whereas in HMS knowledge in one sub-language is equivalent to knowledge in any other sub-language, here it only implies knowledge in more complete sub-languages.[3] Second, because in the present paper knowledge differs across sub-languages, the knowledge modalities "carry" awareness. For example, being aware of formula $k_{\alpha}^{i}\phi$ implies awareness of all propositions in α and is not equivalent to being aware of $k_{\beta}^{i}\phi$. This is not true in HMS, because there is only one knowledge modality. Hence, adapted to the syntax of the present paper, the axiom system of HMS specifies that awareness of $k_{\alpha}^{i}\phi$ only implies awareness of all primitive propositions that generate ϕ, and it is equivalent to awareness of $k_{\beta}^{i}\phi$. This second difference implies, as we show in the following section, that the axiom system of HMS is neither weaker nor stronger than the axiom system of this paper.

Fagin and Halpern [1988] provide the first model of unawareness and introduce an explicit awareness operator, as is the case with the present paper. Modica and Rustichini [1994, 1999], Dekel et al. [1998] and HMS define awareness in terms of knowledge. Both HMS and Halpern and Rêgo [2008] provide sound and complete axiomatizations of Heifetz et al. [2006], hence they are equivalent. Moreover, they are multi agent generalizations of Modica and Rustichini [1999] and Halpern [2001], respectively, which are also equivalent. HMS is also equivalent to a multi agent version of a sub-class of unawareness structures described in Fagin and Halpern [1988].[4] Board and Chung [2007] provide a model of unawareness using first order modal logic.

Heifetz et al. [2006], Li [2008] and Galanis [2007] construct set-theoretic models of unawareness using multiple state spaces. On the other hand, Geanakoplos [1989], Ely [1998] and Xiong [2007] employ the standard framework of a unique state space. Dekel et al. [1998] argue that if unawareness satisfies three plausible properties, then a standard state space can only accommodate trivial unawareness.

Games with unawareness are analyzed by Feinberg [2004, 2005], Čopič and Galeotti [2007], Li [2006b], Sadzik [2006], Heifetz et al. [2007], Heifetz et al. [2008b] and Halpern and Rêgo [2006]. Applications with unawareness have been provided by Modica et al. [1998], Ewerhart [2001], Galanis [2008], Filiz-Ozbay [2008], Ozbay [2008], von Thadden and Zhao [2008] and Zhao [2008].

The paper is organized as follows. Section 2 presents the syntax and the axiom system and compares it to that of HMS. In section 3 we define the unawareness structures and in section 4 we construct the canonical structure. Soundness and completeness are demonstrated in section 5.

[3] The syntax of the two papers is not same. However, we are able to map the syntax of HMS to the syntax of the present paper in a natural way, so that the comparison of the axioms is meaningful.

[4] For more details on the relationships between these papers, see HMS and Halpern and Rêgo [2008].

2 Syntax and axiom system

Let X be the set of primitive propositions, and let I be the set of individuals. The syntax we use involves the usual modalities $\neg$, $\wedge$ and the unusual modalities k^i_α and a^i_α, where $\alpha \subseteq X$. That is, instead of the "objective" knowledge and awareness modalities k^i and a^i, we introduce one for each subset α of the set of primitive propositions.

Given a sequence of primitive propositions and modalities, ϕ, let $Pr(\phi)$ be the set of primitive propositions contained in ϕ.[5] More precisely,

- $Pr(\top) = \emptyset$,
- $Pr(x) = \{x\}$, for $x \in X$,
- $Pr(\neg\phi) = Pr(\phi)$,
- $Pr(\phi \wedge \psi) = Pr(\phi) \cup Pr(\psi)$,
- $Pr(k^i_\alpha \phi) = Pr(\phi) \cup \alpha$,
- $Pr(a^i_\alpha \phi) = Pr(\phi) \cup \alpha$.

The set of formulas $\mathcal{L}$ is the smallest set such that:

- $\top$ is a formula,
- every $x \in X$ is a formula,
- if ϕ is a formula, then $\neg\phi$ is a formula,
- if ϕ and ψ are formulas, then $\phi \wedge \psi$ is a formula,
- if ϕ is a formula and $Pr(\phi) \subseteq \alpha \subseteq X$, then $a^i_\alpha \phi$ and $k^i_\alpha \phi$ are formulas.

Call $\mathcal{L}$ the "universal" language. Given a subset $\alpha \in X$, define the sub-language $\mathcal{L}_\alpha := \{\phi \in \mathcal{L} : Pr(\phi) \subseteq \alpha\}$, which consists of the formulas and the knowledge and awareness modalities containing only primitive propositions in α.

Consider the following axiom system.

[5]The definition of Pr suggests that modalities like k^i_α and a^i_α are also considered "primitive propositions". For example, we can have $Pr(k^i_\alpha \phi) = \alpha \supsetneq Pr(\phi)$. HMS also define a Pr function, but their definition is different. We elaborate on the differences in the next section.

- All substitution instances of valid formulas of Propositional Calculus including the formula $\top$, (PC),
- the inference rule *Modus Ponens*:

$$\frac{\phi, \phi \rightarrow \psi}{\psi}, \tag{MP}$$

For $Pr(\phi), Pr(\psi) \subseteq \beta \subseteq \alpha \subseteq X$,

- the *Axiom of Truth*:

$$k^i_\alpha \phi \rightarrow \phi, \tag{T}$$

- the *Axiom of Positive Introspection*:

$$k^i_\alpha \phi \bigwedge_{x \in \beta} a^i_\alpha x \bigwedge_{y \in \alpha \setminus \beta} \neg a^i_\alpha y \rightarrow k^i_\alpha k^i_\beta \phi, \tag{4}$$

- the *Axiom of Negative Introspection* :

$$a^i_\alpha \phi \bigwedge_{x \in \beta} a^i_\alpha x \rightarrow k^i_\alpha \phi \vee k^i_\alpha (\neg k^i_\beta \phi \wedge a^i_\beta \phi), \tag{5}$$

- the *Propositional Awareness* axioms:

$$\begin{aligned} &1.\ a^i_\alpha \phi \leftrightarrow a^i_\alpha \neg\phi, \\ &2.\ a^i_\alpha \phi \wedge a^i_\alpha \psi \leftrightarrow a^i_\alpha (\phi \wedge \psi), \\ &3.\ a^i_\alpha k^j_\beta \phi \leftrightarrow \bigwedge_{x \in \beta} a^i_\alpha x, \text{ for } j \in I, \\ &4.\ a^i_\alpha a^j_\beta \phi \leftrightarrow \bigwedge_{x \in \beta} a^i_\alpha x, \text{ for } j \in I. \end{aligned} \tag{PA}$$

- the inference rule *RK-Inference*: For all natural numbers $n \geq 1$: If $\phi_1, \phi_2, \ldots, \phi_n$ and ϕ are formulas such that $Pr(\phi) \subseteq U^n_{i=1} Pr(\phi_i) \subseteq \alpha \subseteq X$ then

$$\frac{\phi_1 \wedge \ldots \wedge \phi_n \rightarrow \phi}{k^i_\alpha \phi_1 \wedge \ldots \wedge k^i_\alpha \phi_n \rightarrow k^i_\alpha \phi}. \tag{RK}$$

- For $Pr(\phi) \subseteq \alpha \subseteq \alpha' \subseteq X$,

$$k^i_\alpha \phi \rightarrow a^i_\alpha \phi, \tag{A}$$

$$a^i_\alpha \phi \leftrightarrow a^i_{\alpha'} \phi, \tag{AA}$$

$$k^i_\alpha \phi \rightarrow k^i_{\alpha'} \phi. \tag{KA}$$

Axioms PC and MP are standard and need no explanation. Axioms T,4 and 5 are adapted versions of the familiar axioms $k^i\phi \to \phi$, $k^i\phi \to k^ik^i\phi$ and $k^i\phi \vee k^i\neg k^i\phi$, respectively. The main difference is that these axioms are expressed in a syntax with more than one knowledge modality.

In particular, axiom 4 says that if, within the sub-language generated by primitive propositions in α, the agent knows ϕ and is aware of propositions in β, then he knows that, in the sub-language generated by β, he knows ϕ. Note that the sub-language generated by α cannot express awareness of a primitive proposition outside α. Axiom 5 specifies that being aware of ϕ and all primitive propositions in β implies that either he knows ϕ, or that he knows that, within the sub-language generated by β, he does not know ϕ and he is aware of it.

Axioms PA1 and PA2 are used in Modica and Rustichini [1999] and in HMS but here they are extended for all awareness modalities of all sub-languages. Axiom PA3 specifies that agent i is aware that, within the sub-language generated by β, agent j knows ϕ, if and only if i is aware of all primitive propositions in β. This axiom is similar to the PA3 axiom of HMS: $a^i\phi \leftrightarrow a^ik^j\phi$, for $j \in I$. However, as we discuss in the following section, neither is weaker or stronger than the other. Axiom PA4 has similar intuition.[6] RK-Inference is similar to the RK-Inference rule introduced by HMS. There are two differences. First, the Pr function here is different from the Pr function in HMS. We elaborate on this difference in the following section. Second, the rule here applies to all permitted knowledge modalities. Axiom A specifies that knowledge implies awareness.

The last two axioms specify when awareness and knowledge in one sub-language translate to awareness and knowledge to another sub-language. Axiom AA says that awareness of a formula ϕ in a sub-language generated by α implies awareness of ϕ in all sub-languages which are either more or less complete and can express ϕ. Axiom KA specifies that knowledge of ϕ in a sub-language generated by α implies knowledge of ϕ only in sub-languages which are more complete. This last axiom essentially relaxes the condition that there is one, objective, knowledge modality, that transcends all sub-languages, as in HMS and other papers in the literature.

[6] HMS do not have a PA4 axiom, as they define awareness as $a^i\phi := k^i\phi \vee k^i\neg k^i\phi$, whereas here the only connection between the awareness and knowledge modalities is through the axioms.

2.1 Relation to the axiom system of HMS

In this section we compare the present axiom system with that of HMS. The main difficulty is that the syntax of the two approaches is different. In particular, whereas HMS have one knowledge modality k^i and one awareness modality a^i, the syntax of the present paper contains several knowledge and awareness modalities, k^i_α, a^i_α, one for each subset $\alpha \subseteq X$ of primitive propositions.

We can only have a meaningful comparison if the syntax is the same. This can be achieved if we interpret k^i, a^i in the HMS syntax as the modalities k^i_X, a^i_X, respectively, in the syntax of this paper, where X is the set of all primitive propositions. Moreover, we add to the axiom system of HMS two axioms specifying that all knowledge and awareness modalities are the "same". That is, if $Pr(\phi) \subseteq \alpha \subseteq \alpha' \subseteq X$, we have $k^i_\alpha\phi \leftrightarrow k^i_{\alpha'}\phi$ and $a^i_\alpha\phi \leftrightarrow a^i_{\alpha'}\phi$. HMS define the awareness modality as $a^i\phi := k^i\phi \vee k^i\neg k^i\phi$. We incorporate this definition as an axiom in their axiom system.

Finally, the definition of the function Pr in HMS is different from the definition here. Adapted to the syntax of the present paper, Pr in HMS requires that $Pr(k_\alpha\phi) = Pr(\phi)$, whereas here it requires that $Pr(k_\alpha\phi) = Pr(\phi) \cup \alpha$. This difference matters for the definition of RK-Inference. To distinguish between the two, we denote as Pr' the function Pr of HMS.

Summarizing, the HMS axiom system, adapted to the syntax of the present paper and with the addition of the aforementioned axioms, takes the following form. We denote similar axioms with a $'$.

- Axioms (PC), (MP),

- the *Axiom of Truth*:

$$k^i_X\phi \rightarrow \phi, \qquad \text{(T}'\text{)}$$

- the *Axiom of Positive Introspection*:

$$k^i_X\phi \rightarrow k^i_X k^i_X\phi,, \qquad \text{(4}'\text{)}$$

- the *Propositional Awareness* axioms:

$$\begin{array}{l} 1.\ a^i_X\phi \leftrightarrow a^i_X\neg\phi, \\ 2.\ a^i_X\phi \wedge a^i_X\psi \leftrightarrow a^i_X(\phi \wedge \psi), \\ 3.\ a^i_X\phi \leftrightarrow a^i_X k^j_X\phi, \text{ for } j \in I. \end{array} \qquad \text{(PA}'\text{)}$$

- the inference rule *RK-Inference*: For all natural numbers $n \geq 1$: If $\phi_1, \phi_2, \ldots, \phi_n$ and ϕ are formulas such that $Pr'(\phi) \subseteq U^n_{i=1}Pr'(\phi_i)$ then

$$\frac{\phi_1 \wedge \ldots \wedge \phi_n \rightarrow \phi}{k^i_X\phi_1 \wedge \ldots \wedge k^i_X\phi_n \rightarrow k^i_X\phi}. \qquad \text{(RK}'\text{)}$$

For $Pr(\phi) \subseteq \alpha \subseteq \alpha' \subseteq X$,[7]

$$a^i_\alpha\phi \leftrightarrow a^i_{\alpha'}\phi, \qquad \text{(AA)}$$

$$k^i_\alpha\phi \leftrightarrow k^i_{\alpha'}\phi, \qquad \text{(KA}'\text{)}$$

$$a^i_X\phi \leftrightarrow k^i_X\phi \vee k^i_X\neg k^i_X\phi. \qquad \text{(D)}$$

The first difference between the two axiom systems is that KA′ is relaxed to KA. That is, whereas in HMS there is effectively only one knowledge operator that transcends all sub-languages, in the present axiom system knowledge in one sub-language only implies knowledge in more complete sub-languages.

The second difference is that in the HMS system knowledge and awareness modalities k^i_α and a^i_α do not "carry" any awareness. It is a theorem of the HMS system that being aware of the formula $k^i_\alpha\phi$ is equivalent to being aware of formula $k^i_{\alpha'}\phi$, for any $\alpha' \subseteq X$.[8] This is consistent with the premise that there is effectively only one knowledge modality, k^i. In contrast, in the approach of the present paper knowledge operators "carry" awareness, so that being aware of formula $k^i_\alpha\phi$ does not imply awareness of $k^i_{\alpha'}\phi$. The difference between the two approaches is illustrated by axioms PA3 and PA′3. Although they look similar, it is not the case that one is weaker than the other.

In particular, PA′3 is not a theorem of the current axiom system. To see this, note that if this were the case, then $a^i_X\phi \rightarrow a^i_X k^j_X\phi$ and PA3 would imply that whenever i is aware of a formula ϕ, he is also aware of all primitive propositions in X. For the same reason, PA3 and PA4 are not theorems of the HMS axiom system.[9] As a result, it is not the case that the present axiom system is either weaker or stronger than the HMS axiom system.

The following proposition shows that the remaining axioms are theorems of the HMS system. Let inference rule RK″ be the same as RK but adding the qualification that $Pr'(\phi) \subseteq U^n_{i=1}Pr'(\phi_i)$.

Proposition 1. *Axioms PC, T, 4, 5, PA1, PA2, A, AA, KA and inference rules MP and RK″ are derived from the axiom system of HMS.*

3 Unawareness structures

We first present an overview of the model developed in Galanis [2007]. Consider a complete lattice of disjoint state spaces $\mathcal{S} = \{S_a\}_{a\in A}$ and denote by $\Sigma = \cup_{a\in A}S_a$ the union of these state spaces. A state ω is an element of some state space S. Let S^* be the most complete state space, the join of all state spaces in $\mathcal{S}$.

Let $\preceq$ be a partial order on $\mathcal{S}$. For any S, $S' \in \mathcal{S}$, $S \preceq S'$ means that S' is more expressive than S. Moreover, there is a surjective projection $r^{S'}_S : S' \rightarrow S$. Projections are required to commute. If $S \preceq S' \preceq S''$ then $r^{S''}_S = r^{S'}_S \circ r^{S''}_{S'}$. If $\omega \in S'$, denote $\omega_S = r^{S'}_S(\omega)$ and $\omega_{S''} = \{\omega_1 \in S'' : r^{S''}_{S'}(\omega_1) = \omega\}$. If $B \subseteq S'$, let $B_S = \{\omega_S : \omega \in B\}$ be the restriction of event B to a less expressive state space S and

[7]Note that we use the Pr function, not the Pr' one, because we want the two axioms to hold for all knowledge and awareness modalities that can express formula ϕ.

[8]This is derived from the Propositional Awareness axioms, lemma 1 in HMS, and the definition of Pr'.

[9]Another difference is between inference rules RK and RK′, because Pr is not equivalent to Pr'. Pr specifies that k^i_α, a^i_α carry awareness, so that $Pr(k^i_\alpha\phi) = Pr(\phi) \cup \alpha$, whereas Pr' specifies that they do not, so that $Pr(k^i_\alpha\phi) = Pr(\phi)$.

let $B_{S''} = \bigcup\{\omega_{S''} : \omega \in B\}$ be its enlargement to a more expressive state space S''. Let $g(S) = \{S' : S \preceq S'\}$ be the collection of state spaces that are at least as expressive as S. For a set $B \subseteq S$, denote by $B^{\uparrow} = \bigcup_{S' \in g(S)} (r_S^{S'})^{-1}(B)$ the enlargements of B to all state spaces which are at least as expressive as S.

Consider a possibility correspondence $P^i : \Sigma \to 2^{\Sigma}$ with the following properties:

(0) Confinedness: If $\omega \in S$ then $P^i(\omega) \subseteq S'$ for some $S' \preceq S$.

(1) Generalized Reflexivity: $\omega \in (P^i(\omega))^{\uparrow}$ for every $\omega \in \Sigma$.

(2) Stationarity: $\omega' \in P^i(\omega)$ implies $P^i(\omega') = P^i(\omega)$.

(3) Projections Preserve Awareness: If $\omega \in S'$, $\omega \in P^i(\omega)$ and $S \preceq S'$ then $\omega_S \in P^i(\omega_S)$.

(4) Projections Preserve Ignorance: If $\omega \in S'$ and $S \preceq S'$ then $(P^i(\omega))^{\uparrow} \subseteq (P^i(\omega_S))^{\uparrow}$.

The setting is the same with that of Heifetz et al. [2006]. The first difference is that we completely take out their axiom Projections Preserve Knowledge: If $S \preceq S' \preceq S''$, $\omega \in S''$ and $P^i(\omega) \subseteq S'$ then $(P^i(\omega))_S = P^i(\omega_S)$. Justification and examples for this omission are provided in Galanis [2007]. The two other differences concern the definitions of an event and those of knowledge and awareness.

3.1 Events, awareness and knowledge

An *event* E is a subset of some (necessarily unique) state space $S \in \mathcal{S}$. The negation of E, denoted by $\neg E$, is the complement of E with respect to S. Denote the complement of S by $\emptyset_S$. Let $\mathcal{E} = \{E \subseteq S : S \in \mathcal{S}\}$ be the collection of all events. For each event E, let $S(E)$ be the state space of which it is a subset. An event E "inherits" the expressiveness of the state space of which it is a subset. Hence, we can extend $\preceq$ to a partial order $\preceq_0$ on $\mathcal{E}$ in the following way: $E \preceq_0 E'$ if and only if $S(E) \preceq S(E')$. Abusing notation, we write $\preceq$ instead of $\preceq_0$.

Before defining knowledge, we need to define awareness. For any event E, for any state space S such that $E \preceq S$, define

$$A_S^i(E) = \{\omega \in S : E \preceq P^i(\omega)\}$$

to be the event which describes, with the vocabulary of S, that the agent is aware of event E. The condition $E \preceq S$ imposes that only a state space rich enough to describe E, can also describe the agent's awareness of E. The agent is aware of an event if his possibility resides in a state space that is rich enough to express event E. Unawareness is defined as the negation of awareness. More formally, the event $U_S^i(E)$ describes, with the vocabulary of S, that the agent is unaware of E:

$$U_S^i(E) = \neg A_S^i(E).$$

Let $\Omega^i : \Sigma \to \mathcal{S}$ be such that for any $\omega \in \Sigma$, $\Omega^i(\omega) = S$ if and only if $P^i(\omega) \subseteq S$. $\Omega^i(\omega)$ denotes the agent's state space at ω. An agent knows an event E if he is aware of it and in all the states he considers possible, E is true. Formally, for any event E and for any state space S such that $E \preceq S$, define

$$K_S^i(E) = \{\omega \in A_S^i(E) : P^i(\omega) \subseteq E_{\Omega^i(\omega)}\}.$$

An unawareness structure is defined to be, as in HMS, the tuple

$$\underline{\Sigma} = \left\langle (S_\alpha)_{\alpha \in A}, \left(r_{S_\beta}^{S_\alpha}\right)_{S_\beta \preceq S_\alpha}, \left(P^i\right)_{i \in I} \right\rangle.$$

4 The canonical structure

Recall that, given a subset $\alpha \in X$, $\mathcal{L}_\alpha = \{\phi \in \mathcal{L} : Pr(\phi) \subseteq \alpha\}$ is the sub-language generated by the set α of primitive propositions. Given $\alpha \subseteq X$, define Ω_α to be the set of maximally consistent sets ω_α of formulas in $\mathcal{L}_\alpha$. Let $\Omega = \cup_{\alpha \subseteq X} \Omega_\alpha$ be the collection of all state spaces and define $\Omega_\beta \preceq \Omega_\alpha$ whenever $\beta \subseteq \alpha$. If $\Omega_\beta \preceq \Omega_\alpha$ then the projection $r_\beta^\alpha : \Omega_\alpha \to \Omega_\beta$ is defined as $r_\beta^\alpha(\omega) := \omega \cap \mathcal{L}_\beta$. From proposition 3 and remark 2 of HMS, the projection r_β^α is well defined and surjective, and $\alpha \supseteq \beta \supseteq \gamma$ implies $r_\gamma^\alpha = r_\gamma^\beta \circ r_\beta^\alpha$.

Given a formula ϕ and a subset $\alpha \supseteq Pr(\phi)$, $[\phi]_{\Omega_\alpha} := \{\omega \in \Omega_\alpha : \phi \in \omega\}$ is an event, as it is a subset of state space Ω_α.

Definition 1. *For $\omega \in \Omega_\alpha$, $\alpha \subseteq X$ and $i \in I$, set* $P^i(\omega) := \{\omega' \in \Omega :$ *For every formula* ϕ
i) $k^i_\alpha \phi \in \omega$ *implies* $\phi \in \omega'$
ii) $a^i_\alpha \phi \in \omega$ *iff* $(\phi \in \omega'$ *or* $\neg\phi \in \omega')\}$.[10]

Proposition 2. *For every $i \in I$ and $\omega \in \Sigma$, $P^i(\omega)$ is nonempty and satisfies properties 0-4.*

Corollary 1. *The tuple*

$$\underline{\Omega} = \left\langle (\Omega_\alpha)_{\alpha \subseteq X}, \left(r^\alpha_\beta\right)_{\beta \subseteq \alpha \subseteq X}, \left(P^i\right)_{i \in I} \right\rangle,$$

is an unawareness structure.

Moreover, as the following lemma shows, knowledge and awareness can interchangeably be described syntactically or as an event.

Lemma 1. *Suppose that $\phi \in \mathcal{L}$ and $Pr(\phi) \subseteq \beta \subseteq \alpha \subseteq X$. Then,*

$$[\neg\phi]_{\Omega_\alpha} = \neg[\phi]_{\Omega_\alpha},$$

$$[\phi \wedge \psi]_{\Omega_\alpha} = [\phi]_{\Omega_\alpha} \cap [\psi]_{\Omega_\alpha},$$

$$[k^i_\beta \phi]_{\Omega_\alpha} = \left(K^i_{\Omega_\beta}([\phi]_{\Omega_{Pr(\phi)}})\right)_{\Omega_\alpha},$$

$$[a^i_\beta \phi]_{\Omega_\alpha} = \left(A^i_{\Omega_\beta}([\phi]_{\Omega_{Pr(\phi)}})\right)_{\Omega_\alpha}.$$

Given a formula ϕ, a state $\omega \in \Omega_\alpha$ contains a sequence of knowledge modalities $k^i_{\alpha'}\phi$, where α' is such that $Pr(\phi) \subseteq \alpha' \subseteq \alpha$. Which one of the knowledge modalities is the "true" description of i's knowledge of ϕ? This depends on i's awareness. If ω specifies that i is aware only of primitive propositions in $\alpha' \subseteq \alpha$, then his sub-language is $\Omega_{\alpha'}$ and he knows ϕ if $k^i_{\alpha'}\phi \in \omega$.

It is important to stress that Ω_α, as a description of i's knowledge, can be quite restrictive. The reason is that agent i may be aware of a primitive proposition x which does not belong to α. As a result, sub-language Ω_α is not complete enough to express awareness of x. But more importantly, in that case Ω_α is also not complete enough to express i's knowledge of ϕ as well. In particular, if $\alpha'' = \alpha \cup \{x\}$ then the modality $k^i_{\alpha''}\phi$ is better suited to describe i's knowledge. But it does not belong to the sub-language $\mathcal{L}_\alpha$, so it is not part of any state in Ω_α.

According to the axiom system, a less complete sub-language can only underestimate one's knowledge, not overestimate it. In particular, suppose that $k^i_\alpha \phi \in \omega$ and $\underset{x \in \alpha}{\wedge} a^i_\alpha x \in \omega$ so that i knows ϕ, according to ω. Because of axiom KA, it must be that $k^i_{\alpha''}\phi \in \omega'$ for any $\omega' \in \Omega_{\alpha''}$ that projects to ω, where $\alpha \subset \alpha''$. On the other hand, if $\neg k^i_\alpha \phi \in \omega$ we may have $k^i_{\alpha''}\phi \in \omega'$ or $\neg k^i_{\alpha''}\phi \in \omega'$. Hence, more complete state spaces give a better description of one's knowledge.

Summarizing, it may be that ω' specifies that agent i knows ϕ, whereas the projection of ω' to a lower state space specifies that he does not know ϕ. This is the Awareness Leads to Knowledge property, proposed in Galanis [2007]. The intuition behind this property is that the projection belongs to a state space which is generated by a less complete sub-language, hence containing fewer knowledge modalities $k^i_{\alpha'}\phi$, which may underestimate i's knowledge. This property is not true in HMS, effectively because there is only one knowledge modality, k^i.

5 Soundness and completeness

Recall that $\mathcal{E} = \{E \subseteq S : S \in \mathcal{S}\}$ is the collection of all events and let $\mathcal{E}^\uparrow := \{E^\uparrow : E \in \mathcal{E}\}$ be the collection of *extended events*. A typical element of $\mathcal{E}^\uparrow$ consists of an event $E \subseteq S$ and all of its enlargements to higher state spaces $E_{S'}$, where $S \preceq S'$. For a given set of primitive propositions X, let $v : X \to \mathcal{E}^\uparrow$ be the evaluation function. The extended event $v(x)$ contains all events where the primitive proposition x obtains. An unawareness model is a pair $\underline{\Sigma}^v := (\underline{\Sigma}, v)$. Abusing notation, we write $\underline{\Sigma}$ for an unawareness model, instead of $\underline{\Sigma}^v$. Let $C : \mathcal{S} \to 2^X$ denote which primitive propositions in X occur in state space S. That is, define, for each $S \in \mathcal{S}$, $C(S) := \bigcup\{x \in X : E \in v(x), E \subseteq S\}$. We assume that if $S \neq S'$ then $C(S) \neq C(S')$. Given any set $\alpha \subseteq X$, let $C^{-1}(\alpha) := \bigwedge\{S \in \mathcal{S} : \alpha \subseteq C(S)\}$

[10] Note that $k^i_\alpha \phi, a^i_\alpha \phi$ are defined only if $Pr(\phi) \subseteq \alpha$.

be the least complete state space where all primitive propositions in α occur.

We first specify what it means for a formula ϕ to be defined at a particular state ω.

Definition 2. *For a nonempty set X and a set of players I, let $(\underline{\Sigma}, v)$ be an unawareness model, and let $\omega \in S$ for some $S \in \mathcal{S}$. Then we define by induction on the formation of the formulas in $\mathcal{L}$:*

- $(\underline{\Sigma}, \omega) \mapsto \top$, *for all* $\omega \in \Sigma$,
- $(\underline{\Sigma}, \omega) \mapsto x$, *if* $\omega \in E \in E^{\uparrow} = v(x)$,
- $(\underline{\Sigma}, \omega) \mapsto \phi \wedge \psi$, *if* $(\underline{\Sigma}, \omega) \mapsto \phi$ *and* $(\underline{\Sigma}, \omega) \mapsto \psi$,
- $(\underline{\Sigma}, \omega) \mapsto \neg\phi$, *if* $(\underline{\Sigma}, \omega) \mapsto \phi$,
- $(\underline{\Sigma}, \omega) \mapsto a^i_\alpha \phi$, *if* $Pr(\phi) \subseteq \alpha = C(S')$, $S' \preceq S$, *and* $(\underline{\Sigma}, \omega) \mapsto \phi$,
- $(\underline{\Sigma}, \omega) \mapsto k^i_\alpha \phi$, *if* $Pr(\phi) \subseteq \alpha = C(S')$, $S' \preceq S$, *and* $(\underline{\Sigma}, \omega) \mapsto \phi$.

Definition 3. *Say that a formula ϕ is defined at state $\omega \in S \in \mathcal{S}$ of unawareness model $\underline{\Sigma}$ if $(\underline{\Sigma}, \omega) \mapsto \phi$.*

Note that $k^i_\alpha \phi$, $a^i_\alpha \phi$ are defined at state $\omega \in S$ only if the set of primitive propositions α corresponds to a state space S' ($C(S') = \alpha$) that is less complete than S. In that way, we get a one to one correspondence between the knowledge (awareness) modality k^i_α (a^i_α) and the knowledge (awareness) operator $K^i_{C^{-1}(\alpha)}$ ($A^i_{C^{-1}(\alpha)}$). As the following definition shows, the negation of a formula is true if it is defined but not true.

Definition 4. *For a nonempty set X and a set of players I, let $(\underline{\Sigma}, v)$ be an unawareness model, and let $\omega \in S$ for some $S \in \mathcal{S}$. Then we define by induction on the formation of the formulas in $\mathcal{L}$:*

- $(\underline{\Sigma}, \omega) \models \top$, *for all* $\omega \in \Sigma$,
- $(\underline{\Sigma}, \omega) \models x$, *if* $\omega \in E \in E^{\uparrow} = v(x)$,
- $(\underline{\Sigma}, \omega) \models \phi \wedge \psi$, *if* $(\underline{\Sigma}, \omega) \models \phi$ *and* $(\underline{\Sigma}, \omega) \models \psi$,
- $(\underline{\Sigma}, \omega) \models \neg\phi$, *if* $(\underline{\Sigma}, \omega) \mapsto \neg\phi$ *and not* $(\underline{\Sigma}, \omega) \models \phi$,
- $(\underline{\Sigma}, \omega) \models a^i_\alpha \phi$, *if* $(\underline{\Sigma}, \omega) \mapsto a^i_\alpha \phi$ *and* $\{\omega\}_{C^{-1}(\alpha)} \in A^i_{C^{-1}(\alpha)}([\phi]_{C^{-1}(Pr(\phi))})$,
- $(\underline{\Sigma}, \omega) \models k^i_\alpha \phi$, *if* $(\underline{\Sigma}, \omega) \mapsto k^i_\alpha \phi$ *and* $\{\omega\}_{C^{-1}(\alpha)} \in K^i_{C^{-1}(\alpha)}([\phi]_{C^{-1}(Pr(\phi))})$,

where, given a formula ψ and $S \in \mathcal{S}$ such that $(\underline{\Sigma}, \omega) \mapsto \psi$ for some $\omega \in S$, $[\psi]_S := \{\omega \in S : (\underline{\Sigma}, \omega) \models \psi\}$. Moreover, $\neg[\psi]_S$ is the complement with respect to S.

Theorem 1. *The system of axioms is strongly sound and complete with respect to the class of unawareness models.*

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