Solutions of Strategic Games under Common Belief of Sure-Thing Principle

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Abstract

In this paper we address the issue which solution concept for strategic games is consistent to common belief that each player satisfies the sure-thing principle. Traditional epistemic analysis takes for granted that there is common belief that each player acts according to some expected utility function. Because our presumptions are milder than the traditional ones we are forced to modify the traditional epistemic approach and follow the idea of Morris (1996) to fasten the beliefs of the players to their preferences. One central finding of our paper is that common belief of sure-thing principle plus state-independence characterizes the solution concept proposed by Börgers (1993).

1 Introduction

In standard decision theory the sure-thing principle is generally discussed in context to subjective expected utility theory. For example, consider the axiomization of the expected utility theory of Savage (1954) or Anscombe and Aumann (1963). Because this axiom does not conform with experimental results (e.g. with observations from the experiment of Ellsberg (1961)) decision theorists have come up with alternative utility theories which do not rely on the sure-thing principle or at least rely on weaker versions of it. In our paper we also drop the expected utility hypothesis, but adhere to the sure-thing principle despite of its empirical unreliability . Our goal is to figure out its theoretical relevance for individual decisions under subjective uncertainty, in particular, if this uncertainty arises from a strategic game. Becoming concrete we address the issue which solution concept for strategic games is consistent with a situation at which there is common belief among the players that each player is rational and satisfies the sure-thing principle. Interestingly, if we add the assumption that it is commonly believed among the players that their preferences satisfy the state-independence axiom, then strategies are chosen which survive the elimination process suggested by Börgers (1993). This epistemic characterization for his solution concept differs markedly from his motivation.

In order to detect the solution concept consistent with common belief of sure-thing principle we proceed as follows. In the next section we examine the patterns of individual decisions under uncertainty implied by the preference axioms discussed above. Because our interest focuses on games we presume that the uncertainty of the individual is subjective. In our paper we follow the idea of Savage (1954) and Anscombe and Aumann (1963) and describe subjective uncertainty with a state-space model. Since in this paper we consider only finite games (i.e. games at which the players' strategy sets are finite), our analysis of individual decision problems is restricted to finite choice sets. We specify selection criteria for choices under subjective uncertainty consistent with the sure-thing principle and with the sure-thing principle plus state-independence.

In section 3 we resolve the environment in which the decision maker is stuck. This environment is a strategic game. Various suggestions with regard to the outcome of this game are presented. These solution concepts are reformulations of the selection criteria detected in the single decision theory of section 2 and are not standard in game theory. It turns out that they are variants from the weak undominance concept. Interestingly, the undominance concept which conforms to the selection criterion satisfying the sure-thing principle plus state-independence has been already discovered by Börgers (1993).

In section 4 the strategic game is embedded into an epistemic frame which describes the decision-making of the players. Such frames enable us to give solu-

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tion concepts a decision-theoretic foundation. Traditional epistemic analysis on games takes for granted that that each player acts according to some expected utility function. Because our epistemic assumptions are milder than the traditional ones (additional to the sure-thing principle we require only that the preferences of the players are complete, transitive, nontrivial and monotone) we are forced to modify the traditional epistemic approach. First, the outcomes of the game are interpreted as payoffs rather than index values of utility. Second, our epistemic frame is composed differently. Since we do not presume that the beliefs of the players are representable by probability measures, we have to fasten them to their preference relations. These modifications are not new in epistemic game theory. Epstein and Wang (1996), Epstein (1997) and Di Tillo (2008) discuss such models in depth.

The epistemic frame introduced in section 4 is suitable for exploring the consequences of common belief of sure-thing principle on strategic games. The results of this exploration are presented in section 5. One central result is that the set of outcomes predicted by iterated application of the concept of Börgers (1993) coincides with the set of outcomes which can occur, if there is common belief among the players that each player is rational and each player's preference relation satisfies additional to some mild axioms the sure-thing principle and state-independence. This epistemic characterization of his solution concept differs markedly from his motivation. At the end of section 5 we explain this difference and contrast our approach with his approach. The proofs of the results are presented in the appendix.

2 Choice Structures under Sure-Thing Principle

In a non-cooperative game a player faces uncertainty on the other players' choices. Therefore, breaking down a game into individual decision problems requires to find a model that captures appropriately his uncertainty. Concerning this matter we have to take into account that the player's uncertainty is generally subjective, since in most cases there exists no device that quantifies likelihoods on the other players' choices. For this reason we apply the state space framework proposed by Savage (1954) and Anscombe and Aumann (1963). These authors describe subjective uncertainty by a set of possible states of the world, where each state of the world represents a specific resolution of all relevant uncertain features. In our paper the range of these feature is left open until section 4.

We presuppose that the decision maker under uncertainty (e.g. a player in a game) is endowed with a binary relation $\succsim$, called weak preference relation, on the set $\mathbb{R}^\Omega$ of all real-valued mappings on state space Ω. According to Savage (1954) these mappings are termed acts. The ωth component of act x is denoted by x_ω and indicates the payoff the decision maker gains, when he has chosen act x and state ω occurs. A subset of Ω is termed event and its cardinality is denoted by $\#E$. For any event $E \subseteq \Omega$, x_E denotes the tuple $(x_\omega)_{\omega \in E}$ and $x_{\neg E}$ the tuple $x_{\Omega \setminus E}$. Let $x, y \in \mathbb{R}^\Omega$ and $E \subseteq \Omega$, then $(x_E, y_{\neg E})$ corresponds to the act that yields payoff x_ω if state $\omega \in E$ is realized and payoff y_ω if state $\omega \in \neg E$ is realized. For notational simplicity, we sometimes write ω for event $\{\omega\}$. An act is said to be constant whenever it yields the same payoff at each state. A nonempty finite subset $\mathcal{C}$ of $\mathbb{R}^\Omega$ is termed choice set and shall comprise all acts that are available for the decision maker.

As usual, for $x \succsim y$ it is said that act x is weakly preferred to act y. The strict preference relation $\succ$ is defined by $x \succ y$ whenever $x \succsim y$ and not $y \succsim x$ and the indifference relation $\sim$ is defined by $x \sim y$ whenever $x \succsim y$ and $y \succsim x$. A weak preference relation $\succsim$ on $\mathbb{R}^\Omega$ is called complete, if for any $x, y \in \mathbb{R}^\Omega$ holds that $x \succsim y$ or $y \succsim x$ is true, and transitive, if for any $x, y, z \in \mathbb{R}^\Omega$ holds that $x \succsim y$ and $y \succsim z$ implies $x \succsim z$. It is termed as nontrivial, if there exist acts $x, y \in \mathbb{R}^\Omega$ such that $x \succ y$ holds. In the succeeding analysis we restrict ourselves to the class of complete, transitive and non-trivial weak preference relations on $\mathbb{R}^\Omega$. Henceforth, we denote this class by $\mathscr{R}$. Relevant subclasses of $\mathscr{R}$ will be signed by subscripts.

Under subjective uncertainty the decision maker is confronted with the decision problem to choose one act among the feasible acts. We term his choice as rational, if he weakly prefers the chosen act to any feasible act, and we term the decision maker as rational, if he makes a rational choice for any choice set. Because of the restriction that $\mathcal{C}$ is finite, for any weak preference relations $\succsim \in \mathscr{R}$ each choice set contains a rational choice.

Let $\mathscr{S} \subseteq \mathscr{R}$ be a class of weak preference relations on $\mathbb{R}^\Omega$ and $\mathcal{C}$ be a choice set. An act $x \in \mathcal{C}$ is called *rationalizable* in $\mathcal{C}$ given $\mathscr{S}$, if there exists a weak preference relation $\succsim \in \mathscr{S}$ such that x is a rational choice in $\mathcal{C}$ given $\succsim$. Without any difficulty it can be shown, that every act in any choice set is rationalizable in $\mathscr{R}$. That is, the restriction to the class of complete, transitive and non-trivial weak preference relations does not imply any structure in the choice of a rational decision maker. In this case any choice can be justified as rational. Soon we present subclasses of $\mathscr{R}$ which entail binding criteria for rational choices.

Suppose the decision maker is endowed with a weak preference relation $\succsim \in \mathscr{R}$. A state $\omega \in \Omega$ for which there exist acts $x, y, z \in \mathbb{R}^\Omega$ such that $(x_{\neg\omega}, y_\omega) \succ (x_{\neg\omega}, z_\omega)$ holds, is said to *be considered as possible* by the decision maker. Verbally, a state of the world is considered as possible by the decision maker, if he is not indifferent between all acts which yield different payoffs only at this state. In literature, such states are also called nonnull. The set of nonnull states is termed *possibility set* and is denoted by $P^{\succsim}$ hereafter.

Definition 2.1 *Weak preference relation $\succsim$ on $\mathbb{R}^\Omega$ is called monotone, if for any acts $x, y, z \in \mathbb{R}^\Omega$ with $x_\omega > y_\omega$ for each $\omega \in P^{\succsim}$, it holds $(x_{P^{\succsim}}, z_{\neg P^{\succsim}}) \succ (y_{P^{\succsim}}, z_{\neg P^{\succsim}})$.*

Monotonicity means that an act that yields higher payoffs at each state deemed as possible will always be preferred. Hereafter, the class of all weak preference relations of $\mathscr{R}$ that are monotone is denoted by $\mathscr{R}_m$.

Definition 2.2 *Weak preference relation $\succsim$ on $\mathbb{R}^\Omega$ satisfies sure-thing principle, if for any event $E \subseteq \Omega$ and any acts $v, x, y, z \in \mathbb{R}^\Omega$,*

$$(x_E, z_{\neg E}) \succsim (y_E, z_{\neg E}) \Rightarrow (x_E, v_{\neg E}) \succsim (y_E, v_{\neg E})$$

holds.

Sure-thing principle ensures that personal ranking among two acts is not affected by those states that yield identical payoffs for these acts. Suppose act x agree with act y and act u with act w on states in $\neg E$. If u is identical with x and v is identical with y on each state contained in E, then u is weakly preferred to v, whenever x is weakly preferred to y. Hereafter, the class of all weak preference relations of $\mathscr{R}$ that satisfy the sure-thing principle is denoted by $\mathscr{R}_s$.

It turns out that the choice structure implied by preferences which satisfies monotonicity and sure-thing principle is based on the following notion of dominance.

Definition 2.3 *Let $\mathcal{C} \subseteq \mathbb{R}^\Omega$ and $E \subseteq \Omega$. An act $x \in \mathcal{C}$ is said to be totally weakly dominated in $\mathcal{C}$ given E, if for any state $\omega \in E$ there is an act $y \in \mathcal{C}$ such that y weakly dominates x given E and $y_\omega > x_\omega$.*

Theorem 2.4 *Let $\mathcal{C}$ be some choice set. Then an act x is a rational choice in $\mathcal{C}$ given $\mathscr{R}_{ms}$ if and only if x is not totally weakly dominated in $\mathcal{C}$.*

In the following the implications on the choice structure are analyzed, if a further axiom of the subjective expected utility theory of Savage (1954) is imposed. We integrate the state-independence axiom to our analysis,

Definition 2.5 *Weak preference relation $\succsim$ on $\mathbb{R}^\Omega$ is called state-independent, if for any states $\omega, \tilde{\omega} \in P^{\succsim}$ and any two constant acts $\bar{x}, \bar{y} \in \mathbb{R}^\Omega$ there exists an act $z \in \mathbb{R}$ such that $(\bar{x}_\omega, z_{\neg\omega}) \succsim (\bar{y}_\omega, z_{\neg\omega})$ if and only if $(\bar{x}_{\tilde{\omega}}, z_{\neg\tilde{\omega}}) \succsim (\bar{y}_{\tilde{\omega}}, z_{\neg\tilde{\omega}})$.*

The class of all weak preference relations of $\mathscr{R}$ that are state-independent is denoted by $\mathscr{R}_i$. When weak preference relation $\succsim$ satisfies sure-thing principle additional to state-independence, act z in the above definition can be arbitrarily chosen. In the next theorem we establish that these axioms imply a choice structure which relies on the following concept.

Definition 2.6 *Let $\mathcal{C} \subseteq \mathbb{R}^\Omega$ and $E \subseteq \Omega$. An act $x \in \mathbb{R}^\Omega$ is conditionally weakly dominated in $\mathcal{C}$ given E, if for any event $F \subseteq E$ there exists an act $y \in \mathcal{C}$ such that y weakly dominates x given F.*

Theorem 2.7 *Let $\mathcal{C}$ be some choice set. Then an act x is a rational choice in $\mathcal{C}$ given $\mathscr{R}_{ims}$ if and only if x is not conditionally weakly dominated in $\mathcal{C}$.*

Let us illustrate the content of the above theorems by two concrete decision problems. Consider a state space $\Omega = \{\omega_1, \omega_2\}$ and two choice sets $\mathcal{C}_1$ and $\mathcal{C}_2$. The figure below records the payoffs of their acts. Consider

Choice set $\mathcal{C}_1$

	States	
Act	ω_1	ω_2
u	1	1
v	2	1

Choice set $\mathcal{C}_2$

	States	
Act	ω_1	ω_2
u	1	1
v	2	1
w	0	2

Figure 1: Decision problems under uncertainty

the first decision problem, where the decision maker faces choice set $\mathcal{C}_1$. Note that act u is conditionally weakly undominated in $\mathcal{C}_1$ because there is no act in $\mathcal{C}_1$ that weakly dominates u given event $\{\omega_2\}$. According to theorem 2.7 act u can be rationalized by a monotone, state-independent und sure-thing principle preserving weak preference relation $\succsim$ on $\mathbb{R}^\Omega$. For example, consider weak preference relation $\succsim$ defined by $x \succsim y :\Leftrightarrow x_{\omega_2} \geq y_{\omega_2}$ for all $x, y \in \mathbb{R}^\Omega$. Obviously, such preference relation satisfies those properties and implies $u \succsim v$. Turn to choice set $\mathcal{C}_2$. For any event $E \subseteq \Omega$ we find an act in $\mathcal{C}_2$ that weakly dominates u given E. Whenever E contains state ω_1, act v weakly dominates act u, otherwise act w weakly dominates act u. That means, u is conditionally weakly dominated in $\mathcal{C}_2$. Due to theorem 2.7 act u can not be rationalized in $\mathcal{C}_2$ by any preference relation satisfying above properties. However, u is not totally weakly dominated in $\mathcal{C}_2$, because no act belongs to this choice set that both weakly dominates u and yields at state ω_2 a higher payoff than u. Hence, according to theorem 2.4, act u can be rationalized in the preference class $\mathscr{R}_{ms}$. For example, suppose that the preference relation is represented by utility function $u(x) := x_{\omega_2} - \mathbb{1}_{]-\infty,1]}(x_{\omega_1})(1 - x_{\omega_1})$ for all $x \in \mathbb{R}^\Omega$, where $\mathbb{1}_A$ denotes the indicator function of $A \subseteq \mathbb{R}$. This preference relation is monotone and satisfies sure-thing principle. Furthermore u becomes a rational choice in $\mathcal{C}_2$.

3 Variants of Iterated Undominance

In the preceding section we dealt with decisions under uncertainty. Uncertainty has been described by a state space model, but up to this point it has been left open what is hidden behind a state of the world. In this section we concretize the environment the decision maker faces. We consider a situation at which a group of decision makers interact. Such situations are called games and the rules how this interaction takes place are recorded in the game form. Here we restrict ourselves on the most simple game form, the so called strategic games.

A strategic game Γ is tuple $\Gamma := (S^i, z^i)_{i \in N}$, where N is a non-empty, finite set of players, S^i is a non-empty set of strategies for player i, and $z^i : \times_{j \in N} S^j \to \mathbb{R}$ is player i's payoff function. If S^i is finite for every player $i \in N$, then the strategic game is called finite. Henceforth, we restrict ourselves to finite strategic games. A member s^i of S^i is referred to as strategy for player i and the combination $(s^i)_{i \in N}$ of strategies as strategy profile. Observe payoff function z^i assigns to every strategy profile $(s^i)_{i \in N}$ a real-valued payoff $z^i\left((s^i)_{i \in N}\right)$. We denote the set of all strategy profiles by $S = \times_{i \in N} S^i$ and the set of all profiles listing strategies of players different to i by $S^{-i} = \times_{j \in N \setminus \{i\}} S^j$.

The rules of a game capture all exterior circumstances under which interaction takes place. In traditional game theory, they also constitute the basis upon which a game theorist build his prediction on the outcome of the game. This standard procedure is formally reflected in solution concepts. In general, a solution concept is a algorithm assigning to each game of some class of games a set of strategy profiles which are called solutions of the game. In this section we discuss two variants of the weak dominance concept. These solution concepts are restatements of the selection criteria detected in the previous section.

Definition 3.1 *Consider a strategic game Γ and a player $i \in N$. Let U^i be a nonempty subset of S^i and U^{-i} be a nonempty subset of S^{-i}.*

(a) *Then a strategy $s^i \in U^i$ is called totally weakly dominated in U^i given U^{-i}, if for every $s^{-i} \in U^{-i}$ there exists a strategy $\tilde{s}^i \in U^i$ that both weakly dominates s^i given U^{-i} and satisfies $z^i(\tilde{s}^i, s^{-i}) > z^i(s^i, s^{-i})$.*

(b) *Then a strategy $s^i \in U^i$ is called conditionally weak dominated in U^i given U^{-i}, if s^i is weakly dominated in U^i given any nonempty subset $\tilde{U}^{-i}$ of U^{-i} .*

A strategy s^i that is not totally (conditionally, respectively) weakly dominated in U^i given U^{-i} is termed as totally (conditionally, resp.) weakly undominated in U^i given U^{-i}. Note that what is here called conditional weak dominance is dominance in the sense of Börgers (1993).

Definition 3.2 *Consider a strategic game Γ. The process of iterated elimination of totally (conditionally, resp.) weakly dominated strategies assigns to each player $i \in N$ following inductively determined sequence $(U^i_k)_{k \in \mathbb{N}_0}$ of sets:*

$$U^i_0 := S^i, \qquad U^{-i}_0 := \times_{j \in N \setminus \{i\}} S^j$$

and for every $k \geq 1$

$$\begin{aligned} U^i_k &:= \{s^i \in U^i_{k-1} : s^i \text{ is totally (conditionally, resp.) weakly undominated in } U^i_{k-1} \text{ given } U^{-i}_{k-1}\}, \\ U^{-i}_k &:= \times_{j \in N \setminus \{i\}} U^j_k . \end{aligned}$$

Define $U^i_\infty := \cap_{k \in \mathbb{N}_0} U^i_k$. Strategies belonging to this set are said to be iteratively totally (conditionally, resp.) weakly undominated. Since strategic game Γ is finite, there exists a round l such that $U^i_l = U^i_\infty$ for any player $i \in N$. The solution concept of iterated total (conditional, resp.) weak undominance assigns to each strategic game Γ the nonempty set $U_\infty := \times_{i \in N} U^i_\infty$ of strategy profiles. We end this section by presenting a method to determine iteratively totally (conditionally, resp.) weakly undominated solutions.

Definition 3.3 *Consider a finite strategic game Γ. A family $(X^i)_{i \in N}$ consisting of non-empty subsets of S^i for each $i \in N$ has the total (conditional, resp.) weak undominance property, if for every $i \in N$ each $s^i \in X^i$ is totally (conditionally, resp.) weakly undominated in S^i given X^{-i}.*

Following lemma disclose the relationship between this property and the above elimination process.

Lemma 3.4 *Consider a finite strategic game Γ and player $j \in N$. A strategy $s^j \in S^j$ survives the iterated elimination process presented in definition 3.2, if and only if there exists a family $(X^i)_{i \in N}$ of subsets of*

S^i satisfying the corresponding undominance property presented in definition 3.3 and containing s^j.

Let $\mathscr{U}$ be the finite class of all families having the total (conditional, resp.) weak undominance property in game Γ. Because the family $(U^i_\infty)_{i\in N}$ satisfies this undominance property (see the (only if)-part of the proof of theorem 3.4 in the appendix), $\mathscr{U}$ is nonempty. It is easy to demonstrate that if two families $(X^i)_{i\in N}$ and $(Y^i)_{i\in N}$ belong to $\mathscr{U}$, then their union $(X^i)_{i\in N} \cup (Y^i)_{i\in N} := (X^i \cup Y^i)_{i\in N}$ also belongs to $\mathscr{U}$. In consequence family $(X^i)_{i\in N} := \bigcup\{(Y^i)_{i\in N} : (Y^i)_{i\in N} \in \mathscr{U}\}$ is a member of $\mathscr{U}$. This family is called the largest family in game Γ satisfying the total (conditional, resp.) undominance property. Due to theorem 3.4 any strategy profile of the product of this family is an iteratively totally (conditionally, resp.) weakly undominated solution. On the other side, since $(U^i_\infty)_{i\in N}$ satisfies the above undominance property, each iteratively totally (conditionally, resp.) weak undominated solution is also contained in this product. To sum up, the iteratively undominated solution U_∞ of game Γ is the product of the largest family in Γ which has the undominance property.

As we discussed at the outset of this section the traditional approach in game theory is to deduce from the rules of the game the decisions of the interacting individuals. That is, the prediction on the outcome of the interaction relies only on the external circumstances under which the interaction takes places. Reviewing the different notions of iterated undominance presented in this section we recognize that these procedures follow that approach, since they are related only to items listed in the strategic game form. Clearly, from the point of methodological individualism this approach has to be denied. According to this school of thought outcomes of games like any social phenomenon has to be explained as a result of individual decisions. Obeying these methodological guidelines we are forced to change our perspective on games. Instead of taking the game form as basis of our analysis, we have to plunge into the players and describe their decision-making. Their decisions are the result of the external constraints which they face and their perception about their opponents, that is their conjecture how the opponents will behave. In the next section we introduce a framework which captures these conjectures.

4 Embedding Strategic Games in Epistemic Frames

In order to give solution concepts like those presented in the preceding section a decision-theoretical motivation, we integrate strategic games into state space models. Following the construction of Epstein and Wang (1996) and Di Tillo (2008) each possible state fixes for each player a strategy and a type which in turn is associated with a preference relation on the sets of acts that assign to each state a monetary payoff. Generally, state space models specified in such a way are termed epistemic frames or type space structures.

Definition 4.1 *A type space structure to a strategic game $\Gamma := (S^i, z^i)_{i\in N}$ is a tuple $(T^i, \succsim^i)_{i\in N}$ consisting of the following items:*

- *T^i is a finite set of player i's types, where $\Omega := \times_{i\in N}(S^i \times T^i)$ constitutes the state space and element $\omega \in \Omega$ is called a state of the world.*
- *$\succsim^i$: $T^i \to \mathscr{R}$ is a mapping that assigns to every type $t^i \in T^i$ a complete, transitive and non-trivial weak preference relation $\succsim^i_{t^i}$ on $\mathbb{R}^\Omega$.*

Hereafter t^i_ω, s^i_ω and ω^i are the projections of state ω on T^i, S^i and $S^i \times T^i$, respectively. Let $s^i \in S^i$ be an available strategy of player i in strategic game Γ. If this game is embedded in an epistemic frame, this strategy induces an act $x^{s^i} \in \mathbb{R}^\Omega$, where $x^{s^i}_\omega = z^i(s^i, s^{-i}_\omega)$ for any $\omega \in \Omega$ holds.

Traditional epistemic analysis as surveyed in Battigalli and Bonanno (1999) presumes that a type of a player is determined by a probability measure on the state space. In such a framework the formation of type's belief is straightforward. There an event is believed by a type, whenever he assigns probability one to this event. Because our presumptions do not imply that a type is endowed with a probabilistic belief, we are forced to choose another way for defining type's belief. Here we follow an idea of Morris (1996) and deduce directly type's belief from his preference relation.

Consider an epistemic frame to a strategic game Γ. The possibility set $P^i(\omega)$ of player i at state ω is determined by

$$P^i(\omega) := \{\tilde\omega \in \Omega : \ (y_{\tilde\omega}, x_{\neg\tilde\omega}) \succ^i_{t^i_\omega} (z_{\tilde\omega}, x_{\neg\tilde\omega}) \text{ for some acts } x, y, z \in \mathbb{R}^\Omega\} .$$

States of the world belonging to $P^i(\omega)$ are said to be considered as possible by player i, when ω is the actual state of the world. Because player i's preferences are non-trivial, it holds $P^i(\omega) \neq \emptyset$ for any $\omega \in \Omega$. We say, player i *recognizes himself* at state ω, if

$$\left(x_{\{\omega^i\}\times\Omega^{-i}}, y_{\neg\{\omega^i\}\times\Omega^{-i}}\right) \sim^i_{t^i_\omega} \left(x_{\{\omega^i\}\times\Omega^{-i}}, z_{\{\omega^i\}\times\Omega^{-i}}\right)$$

holds for any acts $x, y, z \in \mathbb{R}^\Omega$. Recognizing himself means that at every state which player i considers as possible his strategy decision as well as his type agree with the actual ones. Hence, at those states his preference relation coincides with the actual one. From the possibility correspondences a set valued function $B^i : 2^\Omega \to \Omega$ is determined by

$$B^i(E) := \{\omega \in \Omega : P^i(\omega) \subseteq E\}$$

for all $E \subseteq \Omega$. This function is called belief operator of player i and assigns to each event the set of states at which all states that player i considers as possible are contained in this event. We say that at state ω player i believes that event E occurs, whenever $\omega \in B^i(E)$ holds. As it will be shown next the belief operator can be expressed directly by the player's preference relation.

Theorem 4.2 *Consider an epistemic frame to a strategic game Γ. Then the belief operator B^i of player i is characterized as follows*

$$B^i(E) = \{\omega \in \Omega : (x_E, y_{\neg E}) \sim^i_{t^i_\omega} (x_E, z_{\neg E}) \text{ for all } x, y, z \in \mathbb{R}^\Omega\} .$$

An event $E \subseteq \Omega$ is mutually believed at state ω, if $\omega \in \cap_{i\in N} B^i(E)$. The correspondence $B : 2^\Omega \to 2^\Omega$ satisfying $B(E) = \cap_{i\in N} B^i(E)$ for any event $E \subseteq \Omega$ is called the mutual belief operator. In words, the mutual belief operator assigns to each event the set of states at which every player believes that event E will be realized. Higher orders of mutual beliefs are defined recursively. Let $B_0 : 2^\Omega \to \Omega$ be an operator satisfying $B_0(E) := E$ for any event $E \subseteq \Omega$. For any $k > 0$ the k-th order belief operator $B_k : 2^\Omega \to \Omega$ is determined by $B_k(E) := B(B_{k-1}(E))$ for any event $E \subseteq \Omega$. Obviously, B_1 corresponds to the mutual belief operator. An event E is said to be commonly believed, if $\omega \in \bigcap_{k=1}^\infty B_k(E)$. The correspondence $B_* : 2^\Omega \to 2^\Omega$ satisfying $B_*(E) = \bigcap_{k=1}^\infty B_k(E)$ for any event E is called the common belief operator. From the common belief operator we construct the common possibility correspondence $P_* : \Omega \to \Omega$, where $P_*(\omega) := \{\tilde\omega \in \Omega : \omega \in \neg B_*(\neg\{\tilde\omega\})\}$. As it will be shown next event E is common believed at state ω, whenever each state that is commonly deemed as possible at state ω belongs to E.

Theorem 4.3 *Consider an epistemic frame to the game Γ, let B_* be the common belief operator and P_* the common possibility correspondence. Then $B_*(E) = \{\omega \in \Omega : P_*(\omega) \subseteq E\}$.*

In the following properties of the common possibility correspondences are deduced, which become useful for the proofs carried out in the next section.

Lemma 4.4 *Consider an epistemic frame to the strategic game Γ and somes states $\omega, \tilde\omega, \hat\omega \in \Omega$.*

(a) *If $\tilde\omega \in P^i(\omega)$ for some player $i \in N$ then $\tilde\omega \in P_*(\omega)$, and the common possibility correspondence is serial $\big($i.e. $P_*(\omega) \neq \emptyset\big)$.*

(b) *The common possibility correspondence is transitive $\big($i.e. $\omega \in P_*(\tilde\omega) \wedge \tilde\omega \in P_*(\hat\omega) \Rightarrow \omega \in P_*(\hat\omega)\big)$, and for any $\tilde\omega \in P_*(\omega)$ it holds $P^i(\tilde\omega) \subseteq P_*(\omega)$ for any player $i \in N$.*

Epistemic analysis on games explores the implications from players' theories about the players' types on the outcome of the game. Usually, we call such theories statements about the world. Let q denote a statement about the world, then $[q]$ denotes the event which consists of all states of the world at which statement q is true. As introduced above $\omega \in B^i([q])$, or equivalently $P^i(\omega) \subseteq [q]$, means that at state ω player i believes that statement q is satisfied, and $\omega \in B_*([q])$, or equivalently $P_*(\omega) \subseteq [q]$, means that at state ω there is common belief among the players that statement q is satisfied. One basic statement in epistemic analysis on games is that the players are rational. A player i is called rational at state ω, if at this state his strategy choice is a rational choice, i.e. if $s^i_\omega \succsim^i_{t^i_\omega} s^i$ holds for any $s^i \in S^i$. Define

$$[\text{RAT}_i] := \{\omega \in \Omega : s^i_\omega \text{ is rational choice given } \succsim^i_{t^i_\omega}\}$$

as the set of states at which player i is rational. Hence, the set

$$[\text{RAT}] := \cap_{i\in N} [\text{RAT}_i]$$

is the set of states at which every player is rational. Further events with which we are tied up in this paper are $[\text{REC}_i]$, $[\text{MON}_i]$, $[\text{IND}_i]$ and $[\text{SURE}_i]$ encasing the set of states at which player i recognizes himself or has monotone, state-independent or sure-thing principle preserving preference relations, respectively. Again, when the subscript i is omitted these events consist of all states at which every player exhibits the respective preference property.

An epistemic frame to a strategic game Γ is said to be consistent to a statement about the world, if the frame contains a state at which this statement is satisfied. A statement about the world characterizes a set $R \subseteq S$ of strategy profiles, if the following two conditions are satisfied:

1. (*Consistency*) If the epistemic frame is consistent with the statement, then at every state satisfying this statement a strategy profile $s \in R$ is realized.
2. (*Existence*) For every $s \in R$ there exists an epistemic frame to the strategic game Γ which contains a state at which this statement is satisfied and at which the strategy profile s is realized.

An epistemic statement is a statement referring to the beliefs of the players. In the next sections we discuss the consequences from common belief of sure thing principle on the outcome of a game. That is, we examine, what set of strategy profiles is characterized by this epistemic statement.

5 Common Belief of Sure-Thing Principle in Strategic Games

In the preceding section we have introduced concepts which are required to carry out epistemic analysis on games without presuming the expected utility hypothesis. In particular, we are ready to figure out the solution concept that is characterized by common belief of rationality, self-awareness, monotonicity and sure-thing principle.

Theorem 5.1 *Consider an epistemic frame to a finite strategic game. Strategy profiles that survive the iterated elimination of totally weakly dominated strategies are characterized by rationality, self-awareness, monotonicity and sure-thing principle and common belief of that.*

Next we give the solution concept that is characterized by common belief of the above properties plus state-independence.

Theorem 5.2 *Consider an epistemic frame to a finite strategic game. Strategy profiles that survive the iterated elimination of conditionally weakly dominated strategies are characterized by rationality, self-awareness, monotonicity and sure-thing principle and common belief of that.*

The last theorem gives an epistemic foundation for iterated application of the dominance concept of Börgers (1993) which differs markedly from the motivation he put forward. He advocates his solution concept by demonstrating that it consists of all game outcomes which are conceivable when players are Bayesian rational. On his reading Bayesian rationality means that the decision maker maximizes the expected value of some Bernoulli utility function, which represents some uniquely determined preference relation on the set of possible payoffs. Moreover, he claims that infinitely iterated application of his dominance concept yields those solutions of the game which remain whenever there is common belief among the players that they are Bayesian rational. In contrast, our epistemic characterization of this iteration does not back on the Bayesian rationality hypothesis. The preference axioms required in theorem 5.2 imply by no means that players' preference relations are representable by an expected utility function. Although these axioms are necessary for an expected utility representation, they are not sufficient. Comparing his and our framework it turns out that the difference between his and our motivation originates from different ways how individual decision-making in games is modeled. Our approach starts with defining preference relations on a set of payoff profiles as opposed to him who starts with preference relations on the set of strategy payoffs.

Our result is in line with the epistemic foundation already given by Lo (2000). There is shown that common belief of eventwise monotonicity (Savage's axiom P3) leads to choices for strategies that survive iterated elimination of conditionally weakly dominated strategies. In our framework this axiom coincides with admissibility (see definition A.1 in the appendix) and (as stated in lemma A.2 in the appendix) is implied by, but not equivalent to monotonicity, state-independence and sure-thing principle. Obviously, our characterization of this elimination process is too loose. Despite of this deficiency we hope that this paper could shed light on the relevance of the sure-thing principle in strategic games.

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Appendix: Proofs of Results

Section 2

Proof of theorem 2.4. (*if*) Suppose act x is not totally weakly dominated in $\mathcal{C}$. Then there exists a state $\omega^* \in \Omega$ such that for every act y that weakly dominates x it holds $y_{\omega^*} = x_{\omega^*}$. In the following a weak preference relation $\succsim \in R_{ms}$ is constructed such that x becomes a rational choice for $\succsim$.

Let $\mathcal{C}'$ be the subset of $\mathcal{C}$ that compromises all acts of $\mathcal{C}$ that at state ω^* yield a higher payoff than x. In case of $\mathcal{C}'$ is empty, then provide the decision maker with a utility function specified by $u(y) := y_{\omega^*}$ for all $y \in \mathbb{R}^\Omega$. Obviously, the preference relation underlying that utility function satisfies the desired axioms and act x becomes a rational choice given $\mathcal{C}$. If $\mathcal{C}'$ is not empty, proceed as follows. Fix E as one of the smallest subsets of Ω containing state ω^* and having the property that no $y \in \mathcal{C}'$ exists which weakly dominates x given E. To any state $\omega \in E \setminus \{\omega^*\}$ assign the set $\mathcal{C}'_\omega$ consisting of all acts $y \in \mathcal{C}'$ satisfying $y_\omega < x_\omega$. For each state $\omega \in E \setminus \{\omega^*\}$ define the scalars

$$\lambda_\omega := \max\{y_\omega : y \in \mathcal{C}'_\omega\}, \quad \lambda^*_\omega := \max\{y_{\omega^*} : y \in \mathcal{C}'_\omega\}$$

and

$$\alpha_\omega := \frac{\lambda^*_\omega - x_{\omega^*}}{x_\omega - \lambda_\omega} \ .$$

Clearly, $\alpha_\omega > 0$ holds. Next an utility function $u : \mathbb{R}^\Omega \to \mathbb{R}$ is specified by

$$u(y) := y_{\omega^*} - \sum_{\omega \in E \setminus \{\omega^*\}} \mathbb{1}_{]-\infty, x_\omega]}(y_\omega)\, (\alpha_\omega (x_\omega - y_\omega))$$

for any $y \in \mathbb{R}^\Omega$, where $\mathbb{1}_A$ is the indicator function for set $A \subseteq \mathbb{R}$. Let $\succsim$ be the weak preference relation underlying that utility function. Without difficulty it can be verified that $P^{\succsim} = E$ holds and that $\succsim$ satisfies monotonicity and sure-thing principle. It remains to demonstrate that x is a rational choice for $\succsim$ in $\mathcal{C}$. Obviously, for any act $y \in \mathcal{C} \setminus \mathcal{C}'$ it holds $u(y) \leq y_{\omega^*} \leq x_{\omega^*} = u(x)$. Otherwise, $F := \{\omega \in E : y_\omega < x_\omega\} \neq \emptyset$ is true and inequality

$$\begin{aligned} & u(x) - u(y) \\ &= x_{\omega^*} - \left(y_{\omega^*} - \sum_{\omega \in F \setminus \{\omega^*\}} \alpha_\omega (x_\omega - y_\omega) \right) \\ &= x_{\omega^*} - y_{\omega^*} + \sum_{\omega \in F \setminus \{\omega^*\}} \frac{\lambda^*_\omega - x_{\omega_*}}{x_\omega - \lambda_\omega} (x_\omega - y_\omega) \\ &\geq x_{\omega^*} - y_{\omega^*} + \sum_{\omega \in F \setminus \{\omega^*\}} \frac{\lambda^*_\omega - x_{\omega_*}}{x_\omega - \lambda_\omega} (x_\omega - \lambda_\omega) \\ &= x_{\omega^*} - y_{\omega^*} + \sum_{\omega \in F \setminus \{\omega^*\}} (\lambda^*_\omega - x_{\omega^*}) \\ &\geq \lambda^*_\omega - y_{\omega^*} \geq 0 \ , \end{aligned}$$

is obtained. Thereby it has been established that x yields the highest utility among the available acts. That is, x is a rational choice in $\mathcal{C}$ given weak preference relation $\succsim$.

(*only if*) Suppose to the contrary that act x is a rational choice in some choice set $\mathcal{C}$ for some weak preference relation $\succsim \in \mathscr{R}_{ms}$, but totally weakly dominated in $\mathcal{C}$. Let $\tilde{\mathcal{C}}$ be a smallest subset of $\mathcal{C}$ in which x is totally weakly dominated. For sake of simplicity, the members of $\tilde{\mathcal{C}}$ are indexed by numbers $j = 1, \ldots, m$. Set $\tilde{\mathcal{C}} = \{y^1, \ldots, y^m\}$ and consider the finite sequence $(y^j)_{j=1}^m$. Since each act y^j weakly dominates x sets

$$E^j := \{\omega \in \Omega : y^j_\omega > x_\omega\}$$

are nonempty. Note that $\cup_{j=1}^m E^j = \Omega$, where $E^j \neq E^k$, whenever j, k are distinct. Assign to each state $\omega \in \Omega$ a set $N_\omega := \{j : \omega \in E^j\}$ and a real number

$$\epsilon_\omega := \frac{1}{2m} \min\{y^j_\omega - x_\omega : j \in N_\omega\} \ .$$

Based on these predefinitions a finite sequence $(z^j)_{j=1}^m$ of acts is constructed. Fix

$$z^1 := \begin{cases} x_\omega \ , & \text{if } y^1_\omega > x_\omega \\ x_\omega + m \cdot \epsilon_\omega \ , & \text{if } y^1_\omega = x_\omega \end{cases}$$

and for $1 < j \leq m$

$$z^j := \begin{cases} x_\omega \ , & \text{if } y^j_\omega > x_\omega \\ x_\omega + (2m-1) \cdot \epsilon_\omega \ , & \text{if } y^j_\omega = x_\omega \\ & \text{and } y^{j-1}_\omega > x_\omega \\ z^{j-1}_\omega - \epsilon_\omega \ , & \text{if } y^j_\omega = x_\omega \\ & \text{and } y^{j-1}_\omega = x_\omega . \end{cases}$$

Obviously, $z^j_\omega = x_\omega$ holds, whenever $\omega \in E^j$, and $z^j_\omega \geq x_\omega + (m+1-j) \cdot \epsilon_\omega$ otherwise. In the following the preference ranking $z^1 \succ z^m$ is established. Since x is a rational choice in $\mathcal{C}$ given some weak preference relation $\succsim \in \mathscr{R}_{ms}$, it holds $x \succsim y^1$. By definition, $z^1 = (z^1_{\neg E^1}, x_{E^1})$. Therefore sure-thing principle implies $z^1 \succsim (z^1_{\neg E^1}, y^1_{E^1})$. Note that for any $\omega \in E^1$ the inequality $y^1_\omega > x_\omega + (2m-1)\epsilon_\omega \geq z^2_\omega$ holds and for any $\omega \in \neg E^1$ the inequality $z^1_\omega > x_\omega + (m-1)\epsilon_\omega \geq z^2_\omega$ is satisfied. By monotonicity $(z^1_{\neg E^1}, y^1_{E^1}) \succ z^2$ is reached and by transitivity $z^1 \succ z^2$. Pick some j, where $1 < j < m$. Since x is a rational choice in $\mathcal{C}$ given $\succsim \in \mathscr{R}_{ms}$, it holds $x \succsim y^j$. Because $z^j = (z^j_{\neg E^j}, x_{E^j})$ sure-thing principle implies $z^j \succsim (z^j_{\neg E^j}, y^j_{E^j})$. In case of $\omega \in E^j$ inequality $y^j_\omega > x_\omega + (2m-1) \cdot \epsilon_\omega \geq z^{j+1}_\omega$ results. If $\omega \in \neg E^{j+1} \cup \neg E^j$, then $z^j_\omega = z^{j+1}_\omega + \epsilon_\omega > z^{j+1}_\omega$ holds. For the remaining case of $\omega \in E^{j+1} \cap \neg E^j$ inequality $z^j_\omega \geq x_\omega + (m-1+j) \cdot \epsilon_\omega > x_\omega = z^{j+1}_\omega$ is satisfied. By monotonicity $(z^j_{\neg E^j}, y^j_{E^j}) \succ z^{j+1}$ is obtained and by transitivity $z^j \succ z^{j+1}$. We have shown that $z^j \succ z^{j+1}$ holds for any $1 \leq j < m$. By transitivity $z^1 \succ z^m$ results. Finally, consider act

$$u := (z^m_{\neg E^m}, y^m_{E^m}) \ .$$

Since x is a rational choice in $\mathcal{C}$ given $\succsim \in \mathscr{R}_{ms}$, it holds $x \succsim y^m$. Note that $z^m = (z^m_{\neg E^m}, x_{E^m})$. Sure-thing principle implies $z^m \succsim u$ and due to transitivity $z^1 \succ u$. When $\omega \in E^m$ holds, then by definition $u_\omega = y_\omega > x_\omega + m \cdot \epsilon_\omega \geq z^1_\omega$ results. When $\omega \in \neg E^m$ holds, then fix

$$k := \max\{j : \omega \in E^j\} \ .$$

Because of $\cup_{j=1}^m E^j = \Omega$ this definition is consistent and $1 \leq k < m$ holds. Clearly, $\omega \in \neg E^j$ is true for each $k < j \leq m$. Thus $z^{k+1}_\omega = x_\omega + (2m-1) \cdot \epsilon_\omega$ and $z^m_\omega = x_\omega + (m+k) \cdot \epsilon_\omega > x_\omega + m \cdot \epsilon_\omega = z^1_\omega$ is obtained. We conclude that $u_\omega > z^1_\omega$ is satisfied for any $\omega \in \Omega$. Because $z^1 \succ u$ holds, a contradiction to monotonicity arises. □

Proof of theorem 2.7. (*if*) For this part of the proof we refer to Börgers (1993). There it is shown that if act x is not conditionally weakly dominated, then there exists a weak preference relation $\succsim$ representable an expected utility function composed of a strictly increasing Bernoulli utility function $u : \mathbb{R} \to \mathbb{R}$ and a probability measure μ on Ω, formally

$$x \succsim y \Leftrightarrow \sum_{\omega \in \Omega} \mu(\omega) u(x_\omega) \geq \sum_{\omega \in \Omega} \mu(\omega) u(y_\omega)$$

for all $x, y \in \mathbb{R}^\Omega$, for which x becomes a rational choice given constraint $\mathcal{C}$. Obviously, such representation implies that $\succsim$ belongs to the class $\mathscr{R}_{ims}$.

(*only if*) For this direction we rely on the following lemma which can be easily checked.

Definition A.1 *Weak preference relation $\succsim$ on $\mathbb{R}^\Omega$ is called admissible, if for any acts $x, y \in \mathbb{R}^\Omega$ holds that if x weakly dominates y given $P^{\succsim}$, then $x \succ y$.*

Lemma A.2 *Each weak preference relation $\succsim \in \mathscr{R}_{ims}$ is admissible.*

Suppose that $\succsim \in \mathscr{R}_{ims}$, but act x is conditionally weakly dominated in $\mathcal{C}$. In particular, there exists an act $y \in \mathcal{C}$ that weakly dominates x on $P^{\succsim}$. We know from lemma A.2 that $\succsim$ is admissible. Therefore $y \succ x$ and thus x is not a rational choice in $\mathcal{C}$. □

Section 3

Proof of lemma 3.4. We restrict ourselves to the case of total weak undominance, because the proof for conditional weak undominance proceeds analogously. (*if*) Let $(U_k^i)_{k\in\mathbb{N}_0}$ for any player $i \in N$ be the sequence of sets that has been generated by the process of iterated elimination of totally weakly dominated strategies. Consider a family $(X^i)_{i\in N}$ of subsets of S^i satisfying the total weak undominance property. Note $X^i \subseteq U_\infty^i$ holds for each $i \in N$, whenever $X^i \subseteq U_k^i$ holds for each $k \in \mathbb{N}_0$ and each $i \in N$. By induction on elimination rounds we will establish the truth of the latter statement. Clearly, $X^i \subseteq U_0^i$ for each $i \in N$. Presume for some elimination round k that $X^i \subseteq U_k^i$ holds for each $i \in N$. Choose some strategy $s^j \in X^j$ of some player j. The total weak undominance property implies that there exists a strategy $s^{-j} \in U_k^{-j}$ for which no strategy $\tilde{s}^j \in S^j$ can be found that both weakly dominates s^j given U_k^{-j} and satisfies $z^j(\tilde{s}^j, s^{-j}) > z^j(s^j, s^{-j})$. It follows that s^j is totally weakly undominated in U_k^j given U_k^{-j}. Hence, $s^j \in U_{k+1}^j$ and we have established that each strategy of X^j is iteratively totally weakly undominated. (*only if*) Again, let $(U_k^i)_{k\in\mathbb{N}_0}$ for each player $i \in N$ be the sequence of sets that has been generated by the process of iterated elimination of totally weakly dominated strategies. It will be established that family $(U_\infty^i)_{i\in N}$ exhibits the total weak undominance property. Suppose to the contrary that there exists a player j and a strategy $s^j \in U_\infty^j$ that is totally weakly dominated in S^j given U_∞^{-j}. That means, for each strategy $s^{-j} \in U_\infty^{-j}$ a strategy $\tilde{s}^j \in S^j$ can be found that both weakly dominates s_j given U_∞^{-j} and fulfils the strict inequality $z_j(\tilde{s}^j, s^{-j}) > z_j(s^j, s^{-j})$. Let $\tilde{S}^j(s^{-j})$ be the set of player's j strategies that weakly dominate strategy s^j given $\tilde{U}_\infty^{-j}$ and yield higher payoffs than s^j given strategy combination s^{-j} is realized by the players different to j. Due to our presumption $\tilde{S}^j(s^{-j})$ is nonempty for any $s^{-j} \in U_\infty^{-j}$. In the following the contradiction will be deduced that neither $\tilde{S}^j(s^{-j}) \cap U_\infty^j \neq \emptyset$ for all strategies $s^{-j} \in U_\infty^{-j}$ nor the opposite is true. Examine initially the first case. Remember that there exists an elimination round l, where $U_l^i = U_\infty^i$ for all $i \in N$. Given $\tilde{S}^j(s^{-j}) \cap U_\infty^j \neq \emptyset$ for all strategies $s^{-j} \in U_\infty^{-j}$, we find for each strategy $s^{-j} \in U_l^{-j}$ a strategy $\tilde{s}_j \in U_l^j$ that weakly dominates s^j given U_l^{-j}, where $z^j(\tilde{s}^j, s^{-j}) > z^j(s^j, s^{-j})$ holds as well. In consequence, strategy s^j is deleted in round $l+1$. But $s^j \notin U_{l+1}^j$ contradicts the presumption $s^j \in U_\infty^j$. Consider the other case, namely that $\tilde{S}^j(s^{-j}) \cap U_\infty^j = \emptyset$ holds for some strategy $s^{-j} \in U_\infty^{-j}$. That is, for some strategy $s^{-j} \in U_\infty^{-j}$ no strategy $\tilde{s}^j \in U_\infty^j$ weakly dominates s^j given U_∞^{-j} and simultaneously satisfies $z^j(\tilde{s}^j, s^{-j}) > z^j(s^j, s^{-j})$. Therefore $k_* := \max\{k \in \mathbb{N}_0 :$ there exists a strategy $\tilde{s}^j \in U_k^j$ weakly dominating s^j given U_∞^{-j} as well as satisfying $z^j(\tilde{s}^j, s^{-j}) > z^j(s^j, s^{-j})\}$ is smaller than l. Select a strategy $\tilde{s}^j \in \tilde{S}^j(s^{-j})$, where $\tilde{s}^j \in U_{k_*}^j \setminus U_{k_*+1}^j$. Obviously, a strategy $\hat{s}^j \in U_{k_*+1}^j$ exists that weakly dominates $\tilde{s}^j$ given $U_{k_*}^{-j}$ and satisfies $z^j(\hat{s}^j, s^{-j}) > z^j(\tilde{s}^j, s^{-j})$. It follows that $\hat{s}^j$ also weakly dominates s^j given $\tilde{U}^{-j}$, where strict inequality $z^j(\hat{s}^j, s^{-j}) > z^j(s^j, s^{-j})$ applies. Consequently, $\hat{s}^j \in \tilde{S}^j(s^{-j})$. But this contradicts the construction of k_*. □

Section 4

Proof of theorem 4.2. The proof is standard and therefore omitted. □

Proof of theorem 4.3. The proof is standard and therefore omitted. □

Proof of lemma 4.4. (a) If $\tilde{\omega} \in P^i(\omega)$, then $P^i(\omega) \nsubseteq \neg\{\tilde{\omega}\}$. That is, $\omega \notin B^i(\neg\{\tilde{\omega}\})$. It follows that $\omega \notin B_*(\neg\{\tilde{\omega}\})$ and therefore $\tilde{\omega} \in P_*(\{\omega\})$. The first part of (a) is proved and, since P^i is serial, it implies that P_* is serial, too. (b) If $\tilde{\omega} \in P_*(\omega)$, then $\omega \notin B_*(\neg\{\tilde{\omega}\})$. Hence, there exists a $k_1 \in \mathbb{N}$ such that $\omega \notin B_{k_1}(\neg\{\tilde{\omega}\})$. Analogously, if $\hat{\omega} \in P_*(\tilde{\omega})$ then there exists a $k_2 \in \mathbb{N}$ such that $\tilde{\omega} \notin B_{k_2}(\neg\{\hat{\omega}\})$. Hence, $B_{k_2}(\neg\{\hat{\omega}\}) \subseteq \neg\{\tilde{\omega}\}$. It follows from the monotonicity of the k_1th-order mutual belief operator that $\omega \notin B_{k_1}(B_{k_2}(\neg\{\hat{\omega}\}))$. Hence, $\omega \notin B_*(\neg\{\hat{\omega}\})$ and therefore $\hat{\omega} \in P_*(\omega)$. Thus the transitivity of P_* is verified. Suppose $\tilde{\omega} \in P_*(\omega)$ and consider some state $\hat{\omega} \in P^i(\tilde{\omega})$. From (a) it follows $\hat{\omega} \in P_*(\tilde{\omega})$ and transitivity of P_* implies $\hat{\omega} \in P^*(\omega)$. Hence, $P^i(\tilde{\omega}) \subseteq P_*(\tilde{\omega})$ holds. □

Section 5

Proof of theorem 5.1. (*Consistency*) Suppose at state ω there is common belief of rationality, self-awareness, monotonicity and sure-thing principle, formally $P_*(\omega) \subseteq [RAT] \cap [REC] \cap [MON] \cap [SURE]$. The set $R^i := \{s^i_{\tilde{\omega}} : \tilde{\omega} \in P_*(\omega)\}$ is the set of strategies of player i that are commonly deemed as possible at state ω. We establish that theses strategies are totally weakly undominated given R^{-i}. Consider an arbitrary strategy $s^i \in R^i$. By construction of R^i there exists a state $\tilde{\omega} \in P_*(\omega)$ such that $s^i = s^i_{\tilde{\omega}}$. Interpreting strategies as acts, self-awareness and rationality of i at state $\tilde{\omega}$ implies that $s^i \succsim^i_{t^i_{\tilde{\omega}}} \tilde{s}^i$ for each $\tilde{s}^i \in S^i$. Note at state $\tilde{\omega}$ player i's preferences are monotone and satisfy sure-thing principle and therefore by theorem 2.4 act s^i is totally weakly undominated given $P^i(\omega)$. Because according to lemma 4.4 $P^i(\tilde{\omega}) \subseteq P_*(\omega)$, act s^i is also totally weakly undominated given $P_*(\omega)$. That means, strategy s^i is not totally weakly dominated given R^{-i}. Until now, we have proved that family $(R^i)_{i\in N}$ satisfies the total weak undominance property. It remains to prove that strategies chosen at state ω are iteratively totally weakly undominated. By lemma 3.4 it suffices to demonstrate that s^i_ω belongs to R^i. Notice that player i judges himself correctly and thereby realizes strategy s^i_ω at each state he deems possible at state ω. Because $P^i(\omega)$ is nonempty and a subset of $P_*(\omega)$, it follows that strategy s^i_ω has to be contained in R^i. (*Existence*) Consider a strategy profile $s \in S$ that survives the iterated elimination of totally weakly dominated strategies. As the discussion after lemma 3.4 revealed there exists a family $(R^i)_{i\in N}$ of subsets of S^i satisfying the total weak undominance property and having $s^i \in R^i$ for each $i \in N$. Specify the type space $T := \times_{i\in N} T^i$ of our epistemic frame by $T^i := R^i$ for all $i \in N$ such that each type of player i is uniquely identified by a strategy of R^i. Remind that each strategy $r^i \in R^i$ is totally weakly undominated given $\Omega^* := R^{-i}$. According to theorem 2.4 we find a monotone and sure-thing principle preserving weak preference relation $\succsim^*_{r^i}$ on $\mathbb{R}^{\Omega^*}$ such that act $r^i \in \mathbb{R}^{\Omega^*}$ becomes a rational choice for type r^i given constraint S^i. Stepwise $\succsim^*_{r^i}$ will be expanded to a weak preference relation $\succsim^i_{r^i}$ that is defined on all real-valued acts having state space $\Omega := \times_{i\in N}(S^i \times R^i)$ as their domain. At first we consider real-valued acts with domain $\Omega^{**} := \{(r^i, r^i)\} \times \Delta_{R^{-i}}$, where $\Delta_{R^{-i}}$ is the diagonal of R^{-i}, formally

$$\Delta_{R^{-i}} := \left\{(r^j, r^j)_{j\in N\setminus\{i\}} : (r^j)_{j\in N\setminus\{i\}} \in R^{-i}\right\} .$$

Let the one-to-one projection from cartesian product Ω^{**} on cartesian product Ω^* be called π and determine a weak preference relation $\succsim^{**}_{r^i}$ defined on Ω^{**} by

$$x \succsim^{**}_{r^i} y :\Leftrightarrow x \circ \pi^{-1} \succsim^*_{r^i} y \circ \pi^{-1} .$$

Finally, the weak preference relation $\succsim^i_{r^i}$ is constructed from $\succsim^{**}_{r^i}$. We define it by

$$x \succsim^i_{r^i} y :\Leftrightarrow x_{\Omega^{**}} \succsim^{**}_{r^i} y_{\Omega^{**}} .$$

Obviously, each type $r^i \in R^i$ of our epistemic frame is endowed with a weak preference relation $\succsim^i_{r^i}$ that belongs to the class $\mathscr{R}_{ms}$. When the actual state belongs to the event Δ_R, then each player recognizes himself and is rational, too. Furthermore, at each state each player considers only states as possible which belongs to Δ_R. Therefore at each state of Δ_R - and in particular at state $(s^i, s^i)_{i\in N}$ - there is common belief that each player recognizes himself, is rational and has monotone and sure-thing principle preserving preferences. □

Proof of theorem 5.2. The proof relies on the same arguments put forward in the proof of theorem 5.1. The only difference is that event $[IND]$ has to be taken into account. Therefore theorem 2.7 is applied instead of theorem 2.4. □

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