Limit Knowledge of Rationality

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Abstract

Epistemic game theory scrutinizes the relationship between knowledge, belief and choice of rational players. Here, the relationship between common knowledge and the limit of higher-order mutual knowledge is studied from a topological point of view. More precisely, the new epistemic operator limit knowledge defined as the topological limit of higher-order mutual knowledge is introduced. We then show that limit knowledge of the specific event rationality can be used for epistemic-topological characterizations of solution concepts in games. As a first step towards this scheme, we construct a game where limit knowledge of rationality appears to be a cogent strict refinement of common knowledge of rationality in terms of solution concepts. More generally, it is shown that for any given game and epistemic model of it satisfying some specific condition, every possible epistemic hypothesis as well as as every solution concept can be characterized by limit knowledge of rationality for some appropriate topology.

1 Introduction

Epistemic game theory scrutinizes the relationship between knowledge, belief, and action of rational game-playing agents. The basic problem addressed is the description of the players' choices in a given game relative to various epistemic assumptions. More precisely, it is attempted to characterize existing game-theoretic solution concepts in terms of epistemic assumptions as well as to propose novel solution concepts by studying the implications of refined or new epistemic hypotheses. Here, we follow the set-based approach to epistemic game theory as introduced and notably developed by Aumann (1976), (1987), (1995), (1999a), (1999b) and (2005).

A central concept in epistemic game theory is common knowledge. It is used in basic background assumptions, such as common knowledge of the game structure, or in epistemic hypotheses, such as common knowledge of rationality, that can be employed to epistemically characterize solution concepts. Originally, the notion has been introduced by Lewis (1969) as a prerequisite for a rule to become a convention. Intuitively, some event is regarded as common knowledge among a set of agents, if everyone knows the event, everyone knows that everyone knows the event, everyone knows that everyone knows that everyone knows the event, etc. Following Lewis's (1969) original proposition, it has become standard to define common knowledge as the infinite intersection, or conjunction, of iterated mutual knowledge claims. Yet, an eminent alternative view of common knowledge as a fixed point also exists. Accordingly, common knowledge of some event is defined as the claim that everyone knows both the event and common knowledge of the event.

The natural question then arises whether these two definitions are equivalent. Barwise (1988) provides a special situation-theoretic model in which the standard and fixed point views of common knowledge do not coincide. Moreover, van Benthem and Sarenac (2005) show the non-equivalence of the two notions in the general framework of epistemic logic with a topological semantics.

A further question that can be addressed concerns the relationship between the standard definition of common knowledge and the infinite sequence of iterated mutual knowledge underlying it. Indeed, Lipman (1994) considers a specific notion of limit such that common knowledge of the particular event rationality is not equivalent to the limit of iterated mutual knowledge of rationality. Here, a topological approach to set-based epistemic game theory is pursued and it is shown that common knowledge is not equivalent to

TARK '09, July 6-8, 2009, California
ISBN: 978-1-60558-560-4...$10.00

the topological limit of the sequence of iterated mutual knowledge. On the basis of this observation the new epistemic operator *limit knowledge* is introduced, and some consequences of limit knowledge of the specific event rationality are scrutinized for games.

2 Common Knowledge

Before common knowledge is defined formally, the set-based framework for interactive epistemology is presented. A so-called Aumann structure $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i \in I})$ consists of a set Ω of possible worlds, which are complete descriptions of the way the world might be, and a possibility partition $\mathcal{I}_i$ of Ω for each agent $i \in I$ representing his information. An event $E \subseteq \Omega$ is defined as a set of possible worlds. For example, the event of it raining in London contains all worlds in which it does rain in London. The cell of $\mathcal{I}_i$ containing the world ω is denoted by $\mathcal{I}_i(\omega)$ and contains all worlds considered possible by i at world ω. In other words, the agent i cannot distinguish between any two worlds ω and ω' that are in the same cell of his partition $\mathcal{I}_i$. Farther, an Aumann structure $\mathcal{A} = (\Omega, (\mathcal{I}_i)_{i \in I})$ is called finite if Ω is finite and infinite otherwise.

The event of agent i knowing E, denoted by $K_i(E)$, is defined as $K_i(E) := \{\omega \in \Omega : \mathcal{I}_i(\omega) \subseteq E\}$. If $\omega \in K_i(E)$, then i is said to know E at world ω. Intuitively, i knows some event E if in all worlds he considers possible E holds. Naturally, the event $K(E) = \bigcap_{i \in I} K_i(E)$ then denotes mutual knowledge of E among the set I of agents. Letting $K^0(E) := E$, m-order mutual knowledge of the event E among the set I of agents is inductively defined by $K^m(E) := K(K^{m-1}(E))$ for all $m > 0$. Accordingly, mutual knowledge can also be denoted as 1-order mutual knowledge. Different higher-order mutual knowledge, also called iterated mutual knowledge, are related by the following lemma:

Lemma 2.1. *For all $m' \geq m \geq 0$, $K^{m'}(E) \subseteq K^m(E)$.*

Proof. The proof is by induction on m'. First of all, suppose $m' = 0$. Then $m = m' = 0$, and obviously $K^{m'}(E) \subseteq K^m(E)$. Now, suppose $m' = p + 1$, for some $p \geq 0$. If $m = m' = p + 1$, then obviously $K^{m'}(E) \subseteq K^m(E)$. If $m = p$, then by definition of the knowledge operator, $K^{m'}(E) = K^{p+1}(E) = K(K^p(E)) \subseteq K^p(E) = K^m(E)$. If $m \leq p$, then by the induction hypothesis, and since the mutual knowledge operator K is monotone with respect to set inclusion, it follows that $K^{m'}(E) = K^{p+1}(E) = K(K^p(E)) \subseteq K(K^m(E)) \subseteq K^m(E)$. □

An event is said to be common knowledge among a set I of agents whenever all m-order mutual knowledge simultaneously hold. The standard definition formalizes this concept as follows.

Definition 2.2. $CK(E) := \bigcap_{m>0} K^m(E)$ *is the event that E is* common knowledge *among the set I of agents.*

Common knowledge of the particular event that all players are rational has been used in epistemic characterizations of solution concepts in games. A well-known result states that common knowledge of rationality implies iterated strict dominance, as provided, for example, by Tan and Werlang (1988) for finite games and involving the standard notion of rationality as subjective expected utility maximization. Below we give an epistemic characterization of pure strategy iterated strict dominance for possibly infinite games and in terms of common knowledge of some weaker rationality. The latter is adapted from Aumann's (1995) knowledge-based extensive form notion which has been argued by Aumann (1995) and (1996) to be simpler and more general than the subjective expected utility maximization one. Iterated strict dominance in pure strategies as well as our modified concept of rationality will serve in the next section to illustrate that our new epistemic operator limit knowledge is capable of cogent implications for games.

Towards this purpose, some standard game-theoretic notation and notions are recalled. A game in normal form $\Gamma = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$ consists of a possibly infinite set of players I, as well as, for each player $i \in I$, a possibly infinite strategy space S_i and a utility function $u_i : \times_{i \in I} S_i \to \mathbb{R}$ that assigns to each strategy profile $(s_i)_{i \in I} \in \times_{i \in I} S_i$ a real number $u_i((s_i)_{i \in I})$ as payoff.

A solution concept $\mathcal{SC}$ is a mapping associating with each game Γ a subset of its strategy profiles $\mathcal{SC}^{\Gamma} \subseteq \times_{i \in I} S_i$. Note that a solution concept thus is a generic method which does not depend on any particular given game.

An epistemic model of a game Γ is an Aumann structure $\mathcal{A}^{\Gamma} = (\Omega, (\mathcal{I}_i)_{i \in I}, (\sigma_i)_{i \in I})$ that additionally specifies for each player $i \in I$ a choice function $\sigma_i : \Omega \to S_i$, connecting the interactive epistemology to the game. The choice function profile $\sigma : \Omega \to \times_{i \in I} S_i$ mapping each world to its corresponding strategy profile is then defined by $\sigma(\omega) = (\sigma_i(\omega))_{i \in I}$. Moreover, it is standard and seems natural to assume that each player knows his own strategy choice, which is formally expressed by requiring each player's choice function σ_i to be measurable with respect to $\mathcal{I}_i$.[1] This so-called measurability assumption has even been denoted as tautologous by Aumann and Brandenburger (1995) who point out that knowing one's own choice is implicit in consciously making a choice.

[1]More precisely, if two worlds ω and ω' are in the same cell of player i's possibility partition, then $\sigma_i(\omega) = \sigma_i(\omega')$.

Next, the adapted notion of rationality used in the sequel is defined.

Definition 2.3. *The event that* player i is rational *is given by*

$$R_i := \bigcap_{s_i \in S_i} (K_i(\{\omega \in \Omega : u_i(s_i, \sigma_{-i}(\omega)) > u_i(\sigma(\omega))\}))^{\complement},$$

and rationality *is the event* $R := \bigcap_{i \in I} R_i$.

In words, a player i is rational whenever for any of his strategies $s_i \in S_i$, he does not know that s_i would yield him higher utility than his actual choice.

Furthermore, given an arbitrary game in normal form, the solution concept iterated strict dominance (ISD) in pure strategies can be defined as follows.

Definition 2.4. *Suppose an arbitrary game in normal form $\Gamma = (I, (S_i)_{i \in I}, (u_i)_{i \in I})$. Let $S_i^0 = S_i$ for all $i \in I$, and let the sequence $(SD^k)_{k \geq 0}$ of strategy profile sets be inductively given by $SD^0 = \times_{i \in I} S_i^0$ and $SD^{k+1} = \times_{i \in I} SD_i^{k+1}$, where $SD_i^{k+1} = SD_i^k \setminus \{s_i \in SD_i^k :$ there exists $s_i' \in SD_i^k$ such that $u_i(s_i, s_{-i}) < u_i(s_i', s_{-i})$, for all $s_{-i} \in SD_{-i}^k\}$, for all $i \in I$. Then, $ISD^\Gamma := \bigcap_{k \geq 0} SD^k$.*

The possible problem of order dependence of ISD, as pointed out, for instance, by Dufwenberg and Stegeman (2002), is avoided by our definition, since at each round, all remaining strictly dominated strategies are eliminated.

The preceding two definitions now permit an epistemic characterization of pure strategy iterated strict dominance in terms of common knowledge of rationality. Note that in Proposition 2.5 below, as well as in all results of Section 3, common knowledge of the structure of the game is taken to be an implicit background assumption.

Proposition 2.5. *Let $\mathcal{A}^\Gamma$ be an epistemic model of an arbitrary game in normal form Γ. Then, $\sigma(CK(R)) \subseteq ISD^\Gamma$.*

Proof. By induction, we prove that $\sigma(K^m(R)) \subseteq SD^{m+1}$, for all $m \geq 0$. It then follows that $\sigma(CK(R)) = \sigma(\bigcap_{m>0} K^m(R)) = \sigma(\bigcap_{m \geq 0} K^m(R)) \subseteq \bigcap_{m \geq 0} \sigma(K^m(R)) \subseteq \bigcap_{m \geq 0} SD^{m+1} = \bigcap_{m>0} SD^m = \bigcap_{m \geq 0} SD^m = ISD^\Gamma$, concluding the proof. First of all, consider $(s_i)_{i \in I} \in \sigma(K^0(R)) = \sigma(R)$. Then, there exists $\omega \in R = \bigcap_{i \in I} R_i$ such that $\sigma(\omega) = (s_i)_{i \in I}$. Hence, by definition of R_i and measurability of σ_i, for all $s_i \in S_i$, there exists $\omega' \in \mathcal{I}_i(\omega)$ such that $u_i(s_i, \sigma_{-i}(\omega')) \leq u_i(\sigma(\omega')) = u_i(\sigma_i(\omega), \sigma_{-i}(\omega'))$. It follows that $\sigma_i(\omega) \in SD_i^1$ for all $i \in I$, thus $\sigma(\omega) \in \times_{i \in I} SD_i^1 = SD^1$. Therefore, $\sigma(K^0(R)) \subseteq SD^1$. Now, assume $\sigma(K^m(R)) \subseteq SD^{m+1}$ for some $m > 0$, and let $(s_i)_{i \in I} \in \sigma(K^{m+1}(R))$. Then, there exists $\omega \in K^{m+1}(R)$ such that $\sigma(\omega) = (s_i)_{i \in I}$. Hence $\mathcal{I}_i(\omega) \subseteq K^m(R)$, and thus by the induction hypothesis, $\sigma(\mathcal{I}_i(\omega)) \subseteq SD^{m+1}$. Besides, since $\omega \in R_i$, for all $s_i \in SD_i^{m+1}$ there exists $\omega' \in \mathcal{I}_i(\omega)$ such that $u_i(s_i, \sigma_{-i}(\omega')) \leq u_i(\sigma(\omega')) = u_i(\sigma_i(\omega), \sigma_{-i}(\omega'))$. Yet since $\sigma(\mathcal{I}_i(\omega)) \subseteq SD^{m+1}$, each $\omega' \in \mathcal{I}_i(\omega)$ induces $\sigma_{-i}(\omega') \in SD_{-i}^{m+1}$, which in turn implies that $\sigma_i(\omega) \in SD_i^{m+2}$ for all $i \in I$, and thus $(s_i)_{i \in I} = \sigma(\omega) \in \times_{i \in I} SD_i^{m+2} = SD^{m+2}$. Therefore, $\sigma(K^{m+1}(R)) \subseteq SD^{m+2}$. □

3 Limit Knowledge

According to the standard definition, common knowledge of an event is the countably infinite intersection of all successive higher-order mutual knowledge of the event. Thence, a natural question to be addressed is to clarify the relationship between common knowledge and the possible limit points of the sequence of higher-order mutual knowledge from a topological point of view. In fact it can be shown that these two concepts are closely related in the case of finite Aumann structures, but do substantially differ for infinite Aumann structures, as illustrated, for instance in Example 3.2 below. The existence of situations in which a unique limit point of the sequence of iterated mutual knowledge differs from common knowledge motivates the following definition of the new epistemic operator limit knowledge.

Definition 3.1. *Let $(\Omega, (\mathcal{I}_i)_{i \in I})$ be an Aumann structure, $\mathcal{T}$ a topology on $\mathcal{P}(\Omega)$, and E an event. If the limit point of the sequence $(K^m(E))_{m>0}$ is unique, then $LK(E) := \lim_{m \to \infty} K^m(E)$ is the event that E is* limit knowledge *among the set I of agents.*

With limit knowledge, a novel operator is proposed that can be employed for epistemic characterizations of existing or new game-theoretic solution concepts, as initiated below. In this context, situations in which limit knowledge differs from common knowledge are of distinguished interest. It can be shown that such situations necessarily involve sequences of iterated mutual knowledge that are strictly shrinking.[2] Note that the expressive power of limit knowledge is severely restricted in case of the discrete topology. Indeed, it can be shown that limit knowledge is not defined if the sequence of iterated mutual knowledge is strictly shrinking, and is equal to common knowledge otherwise.

A possible application of limit knowledge is given by

[2]In the sequel, given some event E, the sequence of iterated mutual knowledge $(K^m(E))_{m>0}$ is said to be *eventually constant* if there exists some index p such that $K^m(E) = K^p(E)$ for all $m \geq p$. Moreover, it is called *strictly shrinking* if $K^{m+1}(E) \subsetneq K^m(E)$ for all $m \geq 0$.

the following example where limit knowledge of rationality indeed appears to be a cogent strict refinement of common knowledge of rationality in terms of solution concepts.

Example 3.2. Consider the Cournot-type game $\Gamma = (I, (S_i)_{i\in I}, (u_i)_{i\in I})$ in normal form with player set $I = \{Alice, Bob, Claire, Donald\}$, strategy sets $S_{Alice} = S_{Bob} = [0,1]$, $S_{Claire} = \{U, D\}$, $S_{Donald} = \{L, R\}$, and utility functions $u_i : S_{Alice} \times S_{Bob} \times S_{Claire} \times S_{Donald} \to \mathbb{R}$ for all $i \in I$, defined as $u_{Alice}(x, y, v, w) = x(1 - x - y)$ and $u_{Bob}(x, y, v, w) = y(1 - x - y)$, as well as $u_{Claire}(x, y, v, w)$ and $u_{Donald}(x, y, v, w)$ given as follows:

Claire \ Donald	L	R
U	(2, 1)	(1, 1)
D	(2, 2)	(2, 3)

for all $(x, y) \neq (\frac{1}{3}, \frac{1}{3})$

Claire \ Donald	L	R
U	(2, 3)	(2, 2)
D	(1, 1)	(2, 1)

for $(x, y) = (\frac{1}{3}, \frac{1}{3})$

Solving the game by iterated strict dominance yields $ISD^\Gamma = \bigcap_{n\geq 0} \left([a_n, b_n]^2 \times \{U, D\} \times \{L, R\}\right) = \{\frac{1}{3}\} \times \{\frac{1}{3}\} \times \{U, D\} \times \{L, R\}$. Yet in this solution, it is possible to further restrict the remaining strategy sets of *Claire* and *Donald* by a weak dominance argument, leaving the singleton set $(ISD + WD)^\Gamma = \{(\frac{1}{3}, \frac{1}{3}, U, L)\}$ as a possible strictly refined solution of the game.[3]

Before turning towards the epistemic model of this game, some preliminary observations are needed. Note that *Alice*'s and *Bob*'s best response functions $b_{Alice} : [0,1] \times \{U, D\} \times \{L, R\} \to [0,1]$ and $b_{Bob} : [0,1] \times \{U, D\} \times \{L, R\} \to [0,1]$ are given by $b_{Alice}(y, v, w) = \frac{1-y}{2}$ and $b_{Bob}(x, v, w) = \frac{1-x}{2}$, respectively. On the basis of these two functions, we now describe an infinite sequence $(s^n_{Alice}, s^n_{Bob})_{n\geq 0}$ of strategy combinations for *Alice* and *Bob* which will be central to the construction of our epistemic model. This sequence is defined for all $n \geq 0$ by induction as follows.

$$\begin{aligned}
(s^0_{Alice}, s^0_{Bob}) &= (0, 1) \\
(s^1_{Alice}, s^1_{Bob}) &= \left(0, \frac{1}{2}\right) \\
(s^{2n+2}_{Alice}, s^{2n+2}_{Bob}) &= \left(\frac{1 - s^{2n+1}_{Bob}}{2}, s^{2n+1}_{Bob}\right) \\
(s^{2n+3}_{Alice}, s^{2n+3}_{Bob}) &= \left(s^{2n+2}_{Alice}, \frac{1 - s^{2n+2}_{Alice}}{2}\right),
\end{aligned}$$

Note that this sequence converges to $(\frac{1}{3}, \frac{1}{3})$.

Next an epistemic model $\mathcal{A}^\Gamma = (\Omega, (\mathcal{I}_i)_{i\in I}, (\sigma_i)_{i\in I})$ is proposed for the game. First of all, the countable set of worlds is given by:

$$\Omega = \{\alpha, \beta, \gamma, \delta, \alpha_0, \beta_0, \gamma_0, \delta_0, \alpha_1, \beta_1, \gamma_1, \delta_1, \alpha_2, \beta_2, \gamma_2, \delta_2, \ldots\}.$$

Second, the possibility partitions are specified as follows:

$$\mathcal{I}_{Alice} = \{\{\alpha, \beta, \gamma, \delta\}\} \cup \{\{\alpha_{2n}, \beta_{2n}, \gamma_{2n}, \delta_{2n}, \alpha_{2n+1}, \beta_{2n+1}, \gamma_{2n+1}, \delta_{2n+1}\} : n \geq 0\}$$

$$\mathcal{I}_{Bob} = \{\{\alpha, \beta, \gamma, \delta\}, \{\alpha_0, \beta_0, \gamma_0, \delta_0\}\} \cup \{\{\alpha_{2n-1}, \beta_{2n-1}, \gamma_{2n-1}, \delta_{2n-1}, \alpha_{2n}, \beta_{2n}, \gamma_{2n}, \delta_{2n}\} : n > 0\}$$

$$\mathcal{I}_{Claire} = \{\{\alpha, \beta\}, \{\gamma, \delta\}\} \cup \{\{\alpha_n, \beta_n\} : n \geq 0\} \cup \{\{\gamma_n, \delta_n\} : n \geq 0\}$$

$$\mathcal{I}_{Donald} = \{\{\alpha, \gamma\}, \{\beta, \delta\}\} \cup \{\{\alpha_n, \gamma_n\} : n \geq 0\} \cup \{\{\beta_n, \delta_n\} : n \geq 0\}$$

Finally, the function $\sigma = (\sigma_{Alice}, \sigma_{Bob}, \sigma_{Claire}, \sigma_{Donald}) : \Omega \to \times_{i\in I} S_i$ assembling all the players' choice functions is defined for all $n \geq 0$ by:

$$\begin{aligned}
\sigma(\alpha) &= (1/3, 1/3, U, L) & \sigma(\alpha_n) &= (s^n_{Alice}, s^n_{Bob}, U, L) \\
\sigma(\beta) &= (1/3, 1/3, U, R) & \sigma(\beta_n) &= (s^n_{Alice}, s^n_{Bob}, U, R) \\
\sigma(\gamma) &= (1/3, 1/3, D, L) & \sigma(\gamma_n) &= (s^n_{Alice}, s^n_{Bob}, D, L) \\
\sigma(\delta) &= (1/3, 1/3, D, R) & \sigma(\delta_n) &= (s^n_{Alice}, s^n_{Bob}, D, R)
\end{aligned}$$

By definition of the sequence $(s^n_{Alice}, s^n_{Bob})_{n\geq 0}$, the two equalities $s^{2n}_{Alice} = s^{2n+1}_{Alice}$ and $s^{2n+1}_{Bob} = s^{2n+2}_{Bob}$ hold for all $n \geq 0$, and therefore our epistemic model satisfies the standard measurability requirement for the players' choice functions.

We now describe the players' rationality in this epistemic model. First, consider *Alice*. Note that she is rational at worlds α, β, γ and δ. Moreover, by construction of the sequence $(s^n_{Alice}, s^n_{Bob})_{n\geq 0}$, if ω is a world such that $(\sigma_{Alice}(\omega), \sigma_{Bob}(\omega)) = (s^{2n}_{Alice}, s^{2n}_{Bob})$ for some $n \geq 0$, then $u_{Alice}(\sigma(\omega)) = u_{Alice}(b_{Alice}(\sigma_{-Alice}(\omega)), \sigma_{-Alice}(\omega)) \geq u_{Alice}(x, \sigma_{-Alice}(\omega))$, for all $x \in S_{Alice}$. Hence, *Alice* is

[3] Formally, given a game Γ, iterated strict dominance followed by weak dominance is defined as $(ISD + WD)^\Gamma = \times_{i\in I}(ISD^\Gamma_i \setminus \{s_i \in ISD^\Gamma_i :$ there exists $s'_i \in ISD^\Gamma_i$ such that $u_i(s_i, s_{-i}) \leq u_i(s'_i, s_{-i})$, for all $s_{-i} \in ISD^\Gamma_{-i}$ and $u_i(s_i, s'_{-i}) < u_i(s'_i, s'_{-i})$ for some $s'_{-i} \in ISD^\Gamma_{-i}\})$.

rational at every world $\omega' \in \mathcal{I}_{Alice}(\omega)$. By definition of $\mathcal{I}_{Alice}$, since each cell contains a world ω such that $(\sigma_{Alice}(\omega), \sigma_{Bob}(\omega)) = (s^{2n}_{Alice}, s^{2n}_{Bob})$ for some $n \geq 0$, it follows that $R_{Alice} = \Omega$. Second, *Bob* is shown not to be rational at every possible world. In fact, his strategies $\sigma_{Bob}(\alpha_0)$, $\sigma_{Bob}(\beta_0)$, $\sigma_{Bob}(\gamma_0)$ and $\sigma_{Bob}(\delta_0)$ all equal 1, which in turn is strictly dominated by any $y \in (0,1)$, thus $\alpha_0, \beta_0\gamma_0, \delta_0 \notin R_{Bob}$. Analogous reasoning as for *Alice* permits to conclude that *Bob* is rational at all remaining worlds. Therefore, $R_{Bob} = \Omega \setminus \{\alpha_0, \beta_0, \gamma_0, \delta_0\}$. Finally, *Claire* and *Donald* are rational at every possible world. Indeed, observe that *Claire* is rational at α, since $\alpha \in \mathcal{I}_{Claire}(\alpha)$ and $u_{Claire}(\sigma(\alpha)) \geq u_{Claire}(D, \sigma_{-Claire}(\alpha))$, while D being her only alternative strategy. As $\beta \in \mathcal{I}_{Claire}(\alpha)$, it follows that *Claire* is also rational at β. Similar arguments hold for *Claire*'s rationality at worlds γ and δ. Analogously, *Claire* is rational at all other possible worlds α_n, β_n, γ_n and δ_n, for all $n \geq 0$. *Donald*'s rationality at each world is obtained in the same manner. Therefore, $R_{Claire} = R_{Donald} = \Omega$ and the event of all players being rational is given by $R = \bigcap_{i \in I} R_i = \Omega \setminus \{\alpha_0, \beta_0, \gamma_0, \delta_0\}$. Consequently, the sequence $(K^m(R))_{m>0}$ is strictly shrinking and the event common knowledge of rationality is given by $CK(R) = \bigcap_{m>0} K^m(R) = \{\alpha, \beta, \gamma, \delta\}$.

Besides, consider the topology on $\mathcal{P}(\Omega)$ given by $\{O \subseteq \mathcal{P}(\Omega) : \{\alpha\} \notin O\} \cup \{\mathcal{P}(\Omega)\}$. Then, the only open neighbourhood of the event $\{\alpha\}$ is $\mathcal{P}(\Omega)$, and all terms of the sequence $(K^m(R))_{m>0}$ are contained in $\mathcal{P}(\Omega)$. Thus $(K^m(R))_{m>0}$ converges to $\{\alpha\}$. Moreover, any singleton $\{F\} \neq \{\{\alpha\}\}$ is open, and since $K^{m+1}(R) \subsetneq K^m(R)$ for all $m > 0$, the sequence $(K^m(R))_{m>0}$ will never remain in the open neighbourhood $\{F\}$ of F from some index onwards. Hence $(K^m(R))_{m>0}$ does not converge to any such event F. Therefore the limit $(K^m(R))_{m>0}$ is unique, and $LK(R) = \lim_{m\to\infty}(K^m(R))_{m>0} = \{\alpha\}$.

Finally, $\sigma(CK(R)) = \{\sigma(\alpha), \sigma(\beta), \sigma(\gamma), \sigma(\delta)\} = \{\frac{1}{3}\} \times \{\frac{1}{3}\} \times \{U, D\} \times \{L, R\} = ISD^\Gamma$, while $\sigma(LK(R)) = \{\sigma(\alpha)\} = \{(\frac{1}{3}, \frac{1}{3}, U, L)\} = (ISD + WD)^\Gamma$. Hence, the solution in accordance with $LK(R)$ is a strict refinement of the solution induced by $CK(R)$.

The preceding example describes a particular topological epistemic model of a given game such that limit knowledge of rationality is a refinement of common knowledge of rationality in terms of solution concepts. In fact, we now generally show that, for any given game and epistemic model of it satisfying the strictly shrinking condition with respect to iterated mutual knowledge of rationality, every possible event as well as every solution concept can be characterized by limit knowledge of rationality for some appropriate topology.

Theorem 3.3. *Let Γ be a normal form and $\mathcal{A}^\Gamma$ an epistemic model of it such that $(K^m(R))_{m>0}$ is strictly shrinking.*

1. *Let E be any event. Then, there exists a topology on $\mathcal{P}(\Omega)$ such that $LK(R) = E$.*
2. *Let $\mathcal{SC}$ be any solution concept. Then, there exists a topology on $\mathcal{P}(\Omega)$ such that $\sigma(LK(R)) \subseteq \mathcal{SC}^\Gamma$.*

Proof.

1. Suppose the topology on $\mathcal{P}(\Omega)$ given by $\mathcal{T} = \{O \subseteq \mathcal{P}(\Omega) : E \notin O\} \cup \{\mathcal{P}(\Omega)\}$. By definition of $\mathcal{T}$, the only open neighbourhood of E is $\mathcal{P}(\Omega)$, and thus $(K^m(R))_{m>0}$ converges to this point. Also, for every $F \neq E$, the singleton $\{F\}$ is open, and by the strictly shrinking condition on $(K^m(R))_{m>0}$, this sequence will never remain in the open neighbourhood $\{F\}$ of F from some index onwards. Hence the sequence $(K^m(R))_{m>0}$ does not converge to F. Therefore, the limit of $(K^m(E))_{m>0}$ is unique, and $LK(R) = \lim_{m\to\infty}(K^m(R))_{m>0} = E$.
2. Consider the event $F = \sigma^{-1}(\mathcal{SC}^\Gamma) = \{\omega \in \Omega : \sigma(\omega) \in \mathcal{SC}^\Gamma\}$. Hence, $\sigma(F) \subseteq \mathcal{SC}^\Gamma$. Now, suppose the topology on $\mathcal{P}(\Omega)$ given by $\mathcal{T}' = \{O \subseteq \mathcal{P}(\Omega) : F \notin O\} \cup \{\mathcal{P}(\Omega)\}$. It then follows that $LK(R) = \lim_{m\to\infty}(K^m(R))_{m>0} = F$. Therefore, $\sigma(LK(R)) = \sigma(F) \subseteq \mathcal{SC}^\Gamma$. □

Epistemic hypotheses being particular events, the above theorem shows that limit knowledge of rationality can be used as a topological foundation for any epistemic hypothesis as well as an epistemic-topological foundation for any solution concept. Observe that Theorem 3.3 can be refined towards equality in the sense that for any epistemic model $\mathcal{A}^\Gamma$ fulfilling its assumptions as well as the additional condition $\sigma(\Omega) \supseteq \mathcal{SC}^\Gamma$, there exists a topology such that $\sigma(LK(R)) = \mathcal{SC}^\Gamma$. In other words, if the epistemic model furnishes a choice function σ that covers all possible strategy profiles given by the solution concept $\mathcal{SC}$, then the choices in accordance with limit knowledge of rationality equal the ones permissible under $\mathcal{SC}$. In this case, limit knowledge of rationality thus provides an exact epistemic-topological foundation for the given solution concept. Farther note that this universal characterization capability of limit knowledge of rationality indispensably requires the strictly shrinking condition to hold. Hence, the expressive power of this epistemic operator is somewhat countered by this significant constraint.

Moreover, the proof of Theorem 3.3 actually provides a generic method to construct a topology such that $\lim_{m\to\infty}(K^m(R))_{m>0} = \sigma^{-1}(\mathcal{SC}^\Gamma)$. The definition of this topology is completely independent from the specific game considered. However, the convergence properties of the sequence $(K^m(R))_{m>0}$ according to this topology do depend on the underlying game. More precisely, while the definition of this topology ensures that $\sigma^{-1}(\mathcal{SC}^\Gamma)$ is always a limit point of the sequence $(K^m(R))_{m>0}$, the uniqueness of this limit point does require the strictly shrinking condition of this sequence to hold, which in turn is related to the structure of the game. Thus the well-definedness and characterization capability of limit knowledge of rationality do depend on the underlying game. Note in this context that it could be of interest to investigate a weakened definition of limit knowledge involving multiple limit points, in order to extend its characterization capability even to situations where the strictly shrinking condition is violated.

4 Discussion

Limit knowledge can be understood as the event which is approached by the sequence of iterated mutual knowledge, according to some notion of closeness between events. In other words, the higher the iterated mutual knowledge, the closer the respective event is to limit knowledge. Yet, limit knowledge should not be seen as any kind of highest iterated mutual knowledge, since it possibly contains worlds that do not belong to any higher-order mutual knowledge.

Generally, epistemic hypotheses revealing some informational mental states of the players are of special interest for epistemic characterizations of solution concepts. Note that limit knowledge of rationality can also be associated with a kind of reasoning pattern of the agents. Indeed, by definition $LK(R) = \lim_{m\to\infty} K^m(R)$, hence it follows that $LK(R)$ holds i.e. the actual world ω belongs to $LK(R)$, if and only if there exists some event E such that both $\omega \in E$ and $E = \lim_{m\to\infty} K^m(R)$, meaning that everyone considers possible a true event which is the topological limit of the sequence $(K^m(R))_{m>0}$. Hence $\omega \in LK(R)$ can be interpreted as everyone considering possible a true event which is eventually topologically indistinguishable from all remaining higher-order mutual knowledge of rationality. In contrast to common knowledge of rationality, the informational mental states of agents in accordance with limit knowledge of rationality do not enable to infer their precise behaviour, but it appears plausible to claim that such mental states significantly influence the agents' subsequent choices.

Theorem 3.3 ensures that several implications of limit knowledge of rationality for epistemic hypotheses as well as for solution concepts in games could be relevant. This epistemic-topological insight can be apprehended from two different angles. A first approach would study possible topological characterizations via limit knowledge of rationality for a given epistemic hypothesis or solution concept. Relevant topological reasoning patterns of the agents in accordance with some given epistemic hypothesis or solution concept could thus be unveiled. Also, seeking conditions for solution concepts which have not yet been epistemically characterized offers an interesting path for further research. Note that Example 3.2 is in line with this first angle, since the involved topology has been chosen in order to make $LK(R)$ correspond precisely to the event that the solution concept $ISD + WD$ is played. Yet, the particular topological characterization of $ISD + WD$ given in Example 3.2 may possibly appear somewhat artificial. The exploration of further topological characterizations of $ISD + WD$ could thus be of interest.

A second approach would derive the epistemic hypotheses or solution concepts in accordance with limit knowledge of rationality, for some given topology. It might be of particular interest to explore the game-theoretic consequences of topologies being defined on the basis of relevant descriptions of the event space or revealing cogent underlying reasoning patterns of the agents. Such topologies could be called *epistemically plausible*. Solution concepts characterizable in this way might be argued to gain in credibility compared to ones that are not. Also, in a more general sense, epistemically plausible topologies could potentially uncover new interesting epistemic hypotheses or solution concepts.

An instance of a epistemically plausible topological foundation for the solution concept n-times strict dominance in pure strategies SD^n is given now. Suppose a game in normal form Γ and some epistemic model $\mathcal{A}^\Gamma$ of it such that the sequence $(K^m(R))_{m>0}$ is strictly shrinking. Given some index $m^* > 0$, consider the topology $\mathcal{T}$ on $\mathcal{P}(\Omega)$ induced by the subbase

$$\Big\{\{K^m(R) : m > 0\}, \{K^m(R) : m > 0\}^{\complement}\Big\} \cup \\ \{\{K^m(R)\} : m < m^*\} \cup \\ \Big\{\{K^{m^*+1}(R), K^{m^*+2}(R), \ldots, K^n(R)\} : n > m^*\Big\}.$$

This topology can be argued to be plausible in the sense that it satisfies the following four properties. First, if E is a term of the sequence $(K^m(R))_{m>0}$ and F is not (or vice versa), then E and F are T_2-separable.[4] Second, if E and F are two distinct terms of $(K^m(R))_{m>0}$ of index strictly smaller than m^*, then E and F are T_2-separable. Third, if E and F are two distinct terms of $(K^m(R))_{m>0}$ of index strictly larger

[4]Given a topological space $(X, \mathcal{T})$, two points in X are called T_2-separable if there exist two disjoint $\mathcal{T}$-open neighbourhoods of these two points.

than m^*, then E and F are T_0-separable but not T_2-separable.[5] Fourth, if $E = K^{m^*}(R)$ and F is any other term of $(K^m(R))_{m>0}$ (or vice versa), then E and F are T_0-separable but not T_2-separable. These properties reflect a particular perception of the event space, where the agents' topological distinction between the first (m^*-1)-order knowledge of rationality is stronger than between the remaining higher-order mutual knowledge. By definition of $\mathcal{T}$, it follows that $LK(R) = K^{m^*}(R)$ and hence $\sigma(LK(R)) \subseteq SD^{m^*+1}$ obtains, by an argument used in the proof of Proposition 2.5. In this sense, $\mathcal{T}$ provides a plausible epistemic-topological characterization of the solution concept SD^n, where $n = m^* + 1$.

5 Conclusion

The topological approach to epistemic game theory initiated here furnishes an enriched framework to interactive epistemology. Similar to the epistemic program that attempts to provide epistemic foundations for solution concepts, a topological approach to epistemic game theory could generate a topological foundation for epistemic hypotheses, as well as an epistemic-topological foundation for solution concepts. Besides, additional insights into the agents' reasoning might be yielded. Farther, the topological methodology used here could be generalized to analyze the relation between any two given operators one of which is defined in topological terms. Possible future work could also focus on studying epistemically plausible topologies and subsequently scrutinizing the implications of limit knowledge of rationality for games.

In a more general sense, it is envisioned to construct a general topological framework for Aumann structures to enrich the epistemic analysis of games. Such an amplification comprises topologies for the state space as well as for the event space. These two components together would then constitute a topological Aumann structure, in which their relationship to each other as well as to epistemic operators and solution concepts could be studied. Also, a general topological framework is capable of phrasing and reflecting the epistemic properties of an interactive situation in topological terms.

[5] Given a topological space $(X, \mathcal{T})$, two points in X are called T_0-separable if there exists a $\mathcal{T}$-open set containing one but not both of these two points.

Acknowledgements

We are highly grateful to Johan van Benthem, Richard Bradley, Adam Brandenburger, Jacques Duparc, Yann Pequignot, and Andrés Perea for illuminating discussions and invaluable comments.

References

R. J. Aumann (1976). Agreeing to Disagree. *Annals of Statistics* 4, 1236–1239.

R. J. Aumann (1987). Correlated Equilibrium as an Expression of Bayesian Rationality. *Econometrica* 55, 1–18.

R. J. Aumann (1995). Backward Induction and Common Knowledge of Rationality. *Games and Economic Behavior* 8, 6–19.

R. J. Aumann (1996). Reply to Binmore. *Games and Economic Behavior* 17, 138–146.

R. J. Aumann (1999a). Interactive Epistemology I: Knowledge. *International Journal of Game Theory* 28, 263–300.

R. J. Aumann (1999b). Interactive Epistemology II: Probability. *International Journal of Game Theory* 28, 301–314.

R. J. Aumann (2005). Musings on Information and Knowledge. *Econ Journal Watch* 2, 88–96.

R. J. Aumann and A. Brandenburger (1995). Epistemic Conditions for Nash Equilibrium. *Econometrica* 63, 1161–1180.

J. Barwise (1988). Three Views of Common Knowledge. In M. Y. Vardi (ed.), *Theoretical Aspects of Reasoning about Knowledge. Proceedings of the Second Conference (TARK 1988)*, 227–243. Morgan Kaufmann.

J. van Benthem and D. Sarenac (2005). The Geometry of Knowledge. In J.-Y. Béziau et al. (ed.), *Aspects of Universal Logic*, 1–31. Centre de Recherches Sémiologiques, Université de Neuchâtel.

M. Dufwenberg and M. Stegeman (2002). Existence and Uniqueness of Maximal Reductions under Iterated Strict Dominance. *Econometrica* 70, 2007–2023.

B. L. Lipman (1994). A Note on the Implications of Common Knowledge of Rationality. *Journal of Economic Theory* 45, 370–391.

T. C. C. Tan and S. R. C. Werlang (1988). The Bayesian Foundation of Solution Concepts of Games. *Journal of Economic Theory* 45, 370–391.

D. K. Lewis (1969). *Convention.* Harvard University Press.