REVISING PREDICTIONS

Giacomo Bonanno
Department of Economics, University of California, Davis, CA 95616-8578, USA

gfbonanno@ucdavis.edu

Abstract

Making a prediction is essentially expressing a belief about the future. It is therefore natural to interpret later predictions as revisions of earlier ones and to investigate the notion of belief revision in this context. We study, both semantically and syntactically, the following *principle of minimum revision of prediction*: "as long as there are no surprises, that is, as long as what actually occurs had been predicted to occur, then everything which was predicted in the past, if still possible, should continue to be predicted, and no new predictions should be added."

1 Introduction

When we make a prediction we select, among the several conceivable future descriptions of the world, those that appear to us to be most plausible. That is, making a prediction is essentially expressing a belief about the future. It is therefore natural to interpret later predictions as revisions of earlier ones and to investigate the notion of belief revision in this context.

The notion of *rational* belief revision is normally identified in the literature with the *conservativity principle* which states that "When changing beliefs in response to new evidence, you should continue to believe as many of the old beliefs as possible" (Harman, 1986, p. 46). This means that if an individual gets new information which is not inconsistent with her previous beliefs, then (1) she has to maintain all the beliefs she previously had and (2) the change in her beliefs should be minimal in the sense that every new proposition that she believes must be deducible from her old beliefs and the new information (see, for example, Gärdenfors, 1988). Applied to our framework, the conservativity principle can be expressed as the following *principle of minimum revision of prediction*:

> "*as long as there are no surprises*, that is, as long as what actually occurs had been predicted to occur, then everything which was predicted in the past, if still possible, should continue to be predicted, and no new predictions should be added."

We provide both a semantic and a syntactic characterization of this principle. We also discuss and characterize a notion of consistency of prediction and its relationship to the principle of minimum revision.

2 The semantics of revision of prediction

Definition 1 *A prediction frame is a triple* $\langle T, \prec, \prec_p \rangle$ *where* T *is a set of instants and* $\prec$ *and* $\prec_p$ *are binary relations on* T *satisfying,* $\forall t_1, t_2, t_3 \in T$,

asymmetry of $\prec$:	*if* $t_1 \prec t_2$ *then* $t_2 \not\prec t_1$,
transitivity of $\prec$:	*if* $t_1 \prec t_2$ *and* $t_2 \prec t_3$ *then* $t_1 \prec t_3$,
backward linearity of $\prec$:	*if* $t_1 \prec t_3$ *and* $t_2 \prec t_3$ *then either* $t_1 = t_2$ *or* $t_1 \prec t_2$ *or* $t_2 \prec t_1$;
$\prec_p$ *subrelation of* $\prec$:	*if* $t_1 \prec_p t_2$ *then* $t_1 \prec t_2$.

The pair $\langle T, \prec \rangle$ is known in temporal logic as *branching time*.[1] Backward linearity of $\prec$ expresses the notion that, while a given moment may have different possible futures, its past is unique. In other words, there is at most one path between any two instants. Our definition expands on the notion of branching time by adding a subrelation $\prec_p$ of the temporal precedence relation $\prec$. We interpret $t_1 \prec t_2$ as saying that t_2 is in the *conceivable future* of t_1, while the interpretation of $t_1 \prec_p t_2$ is that t_2 is in the *predicted future* of t_1. Note that we do not assume that the predicted future of a given moment is a unique history

[1] See, for example, van Benthem (1991) and Burgess (1984).

following that moment (that is, we do *not* require that if $t \prec_p t'$ and $t \prec_p t''$ then either $t' = t''$ or $t' \prec t''$ or $t'' \prec t'$). Furthermore, there is no requirement that the predicted future of a given moment be a *proper* subset of its conceivable future, that is, vague or trivial predictions are allowed. For example, let $T = \{t_1, t_2, t_3, t_4\}$, $\prec = \{(t_1, t_2), (t_1, t_3), (t_1, t_4)\}$ and interpret t_2 as a state where it is sunny, t_3 a state where it rains and t_4 a state where it snows. Then $\prec_p = \prec$ corresponds to the trivial prediction (at time t_1) "tomorrow either it will be sunny or it will rain or it will snow", while $\prec_p = \{(t_1, t_2), (t_1, t_3)\}$ corresponds to the somewhat vague prediction "tomorrow either it will be sunny or it will rain (but it will not snow)" and $\prec_p = \{(t_1, t_2)\}$ corresponds to the sharp prediction "tomorrow it will be sunny".

The dual nature of a prediction frame is reminiscent of the joint treatment (in another branch of modal logic) of knowledge and belief.[2] The set of conceivable future states (the relation $\prec$) can be thought of as what the individual "knows" about the future, while the set of predicted future states (the relation $\prec_p$) represents what she "believes" about the future. In our framework, the conservativity principle for belief revision can be expressed as the following *principle of minimum revision of prediction* (MR), which says that, as long as what actually occurs had been predicted to occur, then everything which was predicted in the past, if still possible, should continue to be predicted, and no new predictions should be added. Formally – letting $C(t) = \{t' \in T : t \prec t'\}$ be the *conceivable* future of t and $P(t) = \{t' \in T : t \prec_p t'\}$ the *predicted* future of t – $\forall t_1, t_2 \in T$,

$$(MR) \quad if \ \ t_1 \prec_p t_2 \ \ and \ \ P(t_1) \cap C(t_2) \neq \emptyset \ \ then \ \ P(t_2) = P(t_1) \cap C(t_2).$$

Imagine that the present moment is t_2. The hypothesis $t_1 \prec_p t_2$ says that the present moment had been predicted at some time in the past, while the hypothesis $P(t_1) \cap C(t_2) \neq \emptyset$ says that some of the predictions made at that time are still possible. The conclusion says that the predictions made at the present moment should be precisely those that were made in the past and are still possible. (MR) can be split into two parts:

[2]See, for example, Battigalli and Bonanno (1997), Halpern (1991), Hintikka (1962), van der Hoek (1993), Lenzen (1978).

$(1) \quad if \;\; t_1 \prec_p t_2, \;\; then \;\; P(t_1) \cap C(t_2) \subseteq P(t_2)$

$(2) \quad if \;\; t_1 \prec_p t_2 \;\; and \;\; P(t_1) \cap C(t_2) \neq \emptyset \;\; then \;\; P(t_2) \subseteq P(t_1) \cap C(t_2).$

(1) says that the set of current predictions must include those that were made in the past and are still possible; thus it is a *non-contraction* requirement. On the other hand, (2) says that no new predictions can be added to those that were made in the past and are still possible; thus it is a *non-expansion* requirement. Clearly, (1) can be written as: $\forall t_1, t_2, t_3 \in T$,

$(MR_1) \quad if \;\; t_1 \prec_p t_2, \; t_1 \prec_p t_3 \;\; and \;\; t_2 \prec t_3 \;\; then \;\; t_2 \prec_p t_3,$

while, given that $P(t_2) \subseteq C(t_2)$ (since $\prec_p \subseteq \prec$), (2) is equivalent to:

$$if \;\; t_1 \prec_p t_2 \;\; and \;\; P(t_1) \cap C(t_2) \neq \emptyset \;\; then \;\; P(t_2) \subseteq P(t_1)$$

which, in turn, can be written as: $\forall t_1, t_2, t_3, t_4 \in T$,

$(MR_2) \quad if \;\; t_1 \prec_p t_2, \; t_1 \prec_p t_3, \; t_2 \prec t_3 \;\; and \;\; t_2 \prec_p t_4 \;\; then \;\; t_1 \prec_p t_4.$

The conjunction of (MR_1) and (MR_2) thus expresses the notion of minimum revision of prediction. In the next section we will discuss the axiomatization of this principle, for which the following two propositions will be useful.

Proposition 2 *In prediction frames,* (MR_1) *is equivalent to the following property:*
$\forall t_1, t_2, t_3, t_4 \in T$,
(MR_a) *if* $t_1 \prec_p t_3, \; t_2 \prec t_3$ *and* $t_4 \prec_p t_2$ *then either* $t_4 \prec t_1$ *or* $t_1 \prec t_4$ *or* $t_2 \prec_p t_3$.

Proof. $(MR_a) \Longrightarrow (MR_1)$: Suppose that $t_1 \prec_p t_2$, $t_1 \prec_p t_3$ and $t_2 \prec t_3$. By asymmetry of $\prec$, $t_1 \not\prec t_1$. Hence it follows from (MR_a) (choosing $t_4 = t_1$) that $t_2 \prec_p t_3$. $(MR_1) \Longrightarrow (MR_a)$: Suppose that $t_1 \prec_p t_3$, $t_2 \prec t_3$ and $t_4 \prec_p t_2$. Since $\prec_p$ is a subrelation of $\prec$, $t_1 \prec t_3$ and $t_4 \prec t_2$. Since $t_1 \prec t_3$ and $t_2 \prec t_3$, by backward linearity of $\prec$ either (1) $t_2 \prec t_1$ or (2) $t_1 = t_2$ or (3) $t_1 \prec t_2$. In case (1), using $t_4 \prec t_2$ and transitivity of $\prec$, we get $t_4 \prec t_1$. In case (2) it follows from $t_4 \prec t_2$ that $t_4 \prec t_1$. In case (3) using $t_4 \prec t_2$ and backward linearity of $\prec$ we get that either $t_4 \prec t_1$ or $t_1 \prec t_4$ or $t_1 = t_4$. If $t_1 = t_4$, then from (MR_1) we get that $t_2 \prec_p t_3$. ■

Proposition 3 *In prediction frames,* (MR_2) *is equivalent to the following property:*
$\forall t_1, t_2, t_3, t_4, t_5 \in T$,
(MR_b) *if* $t_1 \prec_p t_3, \; t_2 \prec t_3, \; t_5 \prec_p t_2$ *and* $t_2 \prec_p t_4$ *then either* $t_5 \prec t_1$ *or* $t_1 \prec t_5$ *or* $t_1 \prec_p t_4$.

Proof. $(MR_b) \Longrightarrow (MR_2)$: Suppose that $t_1 \prec_p t_2$, $t_1 \prec_p t_3$, $t_2 \prec t_3$ and $t_2 \prec_p t_4$. By asymmetry of $\prec$, $t_1 \nprec t_1$. Thus by (MR_b), choosing $t_5 = t_1$, $t_1 \prec_p t_4$. $(MR_2) \Longrightarrow (MR_b)$: Suppose that $t_1 \prec_p t_3$, $t_2 \prec t_3$, $t_5 \prec_p t_2$ and $t_2 \prec_p t_4$. Since $\prec_p$ is a subrelation of $\prec$, $t_1 \prec t_3$ and $t_5 \prec t_2$. From $t_5 \prec t_2$, $t_2 \prec t_3$ and transitivity of $\prec$ we get that $t_5 \prec t_3$. From $t_1 \prec t_3$ and $t_5 \prec t_3$ it follows from backward linearity of $\prec$ that either $t_5 \prec t_1$ or $t_1 \prec t_5$ or $t_1 = t_5$. If $t_1 = t_5$, by (MR_2), $t_1 \prec_p t_4$. ∎

Corollary 4 *In prediction frames, (MR) is equivalent to the conjunction of (MR_1) and (MR_2), which, in turn, is equivalent to the conjunction of (MR_a) and (MR_b).*

While (MR) captures the notion of minimum revision of prediction, which can be interpreted as a requirement of rationality, there are further properties that seem to be natural expressions of the notion of "rational" prediction. We will consider some such properties and their relationship with (MR).

As remarked previously, predictions can be thought of as beliefs about the future. Even if one allows for maximum freedom in the formation of such beliefs (e.g. in 1999 one could not have labeled as "illogical" the belief that the world would come to an end on January 1, 2000) it seems that *some* restrictions of a logical nature ought to be imposed. For example, consider the following statement:

> ‹‹I have never been to Italy, I am not in Italy now and I predict that at some time in the future I will be able to truthfully assert "I have been to Italy"››.

It seems that, on logical grounds, we must require that the person who makes these assertions be willing to state:

> ‹‹I predict that I will be in Italy at some time in the future››.[3]

It will be shown in the next section that this requirement corresponds to the following property ('CP' stands for 'consistency of prediction'): $\forall t_1, t_2, t_3 \in T$,

(CP_1) *if* $t_1 \prec_p t_3$, $t_2 \prec t_3$, $t_1 \neq t_2$ *and* $t_2 \nprec t_1$ *then* $t_1 \prec_p t_2$.

[3] In the first statement the speaker foresees a point in the future where she will look back in time and see herself in Italy. Since at the present moment she is not in Italy, and she has never been to Italy in the past, then she must be foreseeing a future time when she will be in Italy.

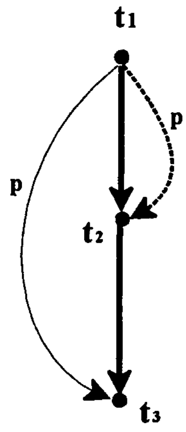	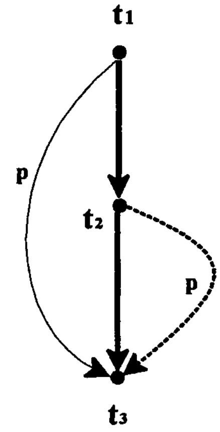
Figure 1a	Figure 1b

Property (CP_1) is shown in Figure 1a, where a thick arrow from from t to t' denotes that $t \prec t'$ (omitting arrows that can be obtained by transitivity) and a thin or dotted arrow with the label 'p' from t to t' denotes that $t \prec_p t'$ (the thin arrow is part of the premise, while the dotted arrow is the conclusion). Note that the thick arrow from t_1 to t_2 (that is, the hypothesis that $t_1 \prec t_2$) is implicit in the antecedent of (CP_1): in fact, $t_1 \prec_p t_3$ implies $t_1 \prec t_3$ (since $\prec_p$ is a subrelation of $\prec$) and the latter, together with the hypothesis that $t_2 \prec t_3$ and $t_1 \neq t_2$ and $t_2 \not\prec t_1$, yields $t_1 \prec t_2$ by backward linearity of $\prec$.

Consider now the following statement:

> ‹‹I predict that at some time in the future I will visit Italy after having visited France. I am not in France now and I have never been to France››.

It seems plausible, on consistency grounds, to require that the individual be willing to state the following:

> ‹‹I can conceive of a time in the future when I will be in France and I will predict a future visit to Italy››.

It will be shown in the next section that this requirement corresponds to the property: $\forall t_1, t_2, t_3 \in T$,

(CP_2) $\quad if\ t_1 \prec_p t_3,\ t_2 \prec t_3,\ t_1 \neq t_2\ and\ t_2 \nprec t_1\ \ then\ \ t_2 \prec_p t_3.$

Property (CP_2) is shown in Figure 1*b* (once again, the thick arrow from t_1 to t_2, that is, the hypothesis that $t_1 \prec t_2$, is implicit in the antecedent).

As the following proposition shows, the above two properties can be condensed into the following: $\forall t_1, t_2, t_3 \in T$,

(CP) $\quad if\ t_1 \prec_p t_3,\ t_1 \prec t_2\ and\ t_2 \prec t_3\ then\ t_1 \prec_p t_2\ and\ t_2 \prec_p t_3.$

It is worth stressing that the requirement expressed by (CP) is imposed within a framework where there is a *unique* path from t_1 to t_3, and t_2 belongs to that path (indeed, the characteristic feature of branching time is that each instant has a unique past history, while the future is open).

The following is a seemingly more general version of (CP):

(CP') If $t_1, ..., t_n \in T$ are such that $t_1 \prec_p t_n$ and $t_i \prec t_{i+1}$, $\forall i = 1, ..., n-1$, then $t_i \prec_p t_{i+1}$ $\forall i = 1, ..., n-1$.

Proposition 5 *In prediction frames, (CP') is equivalent to (CP) which, in turn, is equivalent to the conjunction of (CP_1) and (CP_2).*[4]

Proof. (CP) is the special case of (CP') where $n = 3$. To show that (CP) implies (CP'), let $t_1, ..., t_n \in T$ be such that $t_1 \prec_p t_n$ and $t_i \prec t_{i+1}$, $\forall i = 1, ..., n-1$, with $n > 3$. By transitivity of $\prec$, $t_1 \prec t_{n-1}$. Thus by (CP) $t_{n-1} \prec_p t_n$ and $t_1 \prec_p t_{n-1}$. Thus we have reduced to the case $n-1$. If $n-1 = 3$, the proof is completed by a second application of (CP). If $n-1 > 3$ the argument can be repeated until the sequence is reduced to three elements. To prove the equivalence of (CP) and the conjunction of (CP_1) and (CP_2), it is sufficient to note that, by asymmetry and backward linearity of $\prec$ and the fact that $\prec_p$ is a subrelation of $\prec$, (i) and (ii) below are equivalent: (i) $t_1 \prec_p t_3$, $t_1 \prec t_2$ and $t_2 \prec t_3$, (ii) $t_1 \prec_p t_3$, $t_2 \prec t_3$, $t_1 \neq t_2$ and $t_2 \nprec t_1$. ■

[4] *An alternative characterization of (CP) is given in Bonanno (2001).*

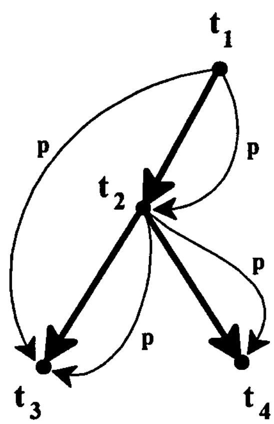

Figure 2*a*

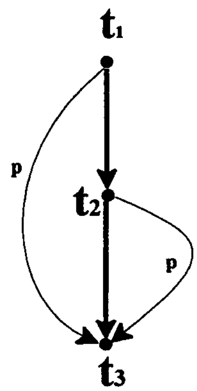

Figure 2*b*

Remark 6 *Consistency (CP) and minimum revision (MR) of prediction are two are independent properties. Figure 2a shows a frame that satisfies (CP) but not (MR) (since $P(t_1) \cap C(t_2) = \{t_3\} \neq C(t_2) = \{t_3, t_4\}$), while Figure 2b shows a frame that satisfies (MR) but not (CP) (since $t_1 \not\prec_p t_2$).*[5]

3 The syntax of revision of prediction

We now turn to the syntax. We take as starting point the temporal logic of branching time (see Burgess, 1984). To the modal operators G and H, we add two more: G_p and H_p. The intended interpretation is:

$G\phi$: "it is going to be the case in every *conceivable* future that ϕ"

$H\phi$: "it has always been the case that ϕ"

$G_p\phi$: "it is going to be the case in every *predicted* future that ϕ"

$H_p\phi$: "at every past instant at which today was predicted it was the case that ϕ".

[5] *It is the non-expansion property (MR_2) that makes the two concepts unrelated. In fact, it is easy to see that (CP_2) is a strengthening of (MR_1) and thus implies it: assume (CP_2) and let the antecedent of (MR_1) be satisfied, that is, let $t_1 \prec_p t_2$, $t_1 \prec_p t_3$ and $t_2 \prec t_3$. Since $\prec_p$ is a subrelation of $\prec$, $t_1 \prec t_2$. Thus by asymmetry of $\prec$, $t_1 \neq t_2$ and $t_2 \not\prec t_1$. Hence, by (CP_2), $t_2 \prec_p t_3$.*

It can also be shown that in prediction frames where $\prec_p$ is transitive (CP) implies (MR). The converse, however, is not true, as the example of Figure 2b shows.

Let $F\phi := \neg G \neg \phi$, $P\phi := \neg H \neg \phi$, $F_p\phi := \neg G_p \neg \phi$ and $P_p\phi := \neg H_p \neg \phi$. Thus the interpretation of $F\phi$ is "at some *conceivable* future instant it will be the case that ϕ", while $F_p\phi$ is interpreted as "at some *predicted* future instant it will be the case that ϕ ". Similarly for $P\phi$ and $P_p\phi$. The logic of prediction frames is obtained by adding the following axiom to branching time logic (corresponding to the property that $\prec_p$ is a subrelation of $\prec$):

(A_p) $G\phi \to G_p\phi$.

Given a prediction frame $\langle T, \prec, \prec_p \rangle$, a *model based on it* is obtained by adding a function $V : S \to 2^T$ (where 2^T denotes the set of subsets of T) that associates with every sentence letter q the set of instants at which q is true. Given a model, an instant t and a formula ϕ, we denote that ϕ is true at t by $t \models \phi$; furthermore, $||\phi||$ denotes the truth set of ϕ, that is, $||\phi|| = \{t \in T : t \models \phi\}$. The usual rules apply, in particular, $t \models G\phi$ if and only if $t' \models \phi$ for all t' such that $t \prec t'$ and $t \models G_p\phi$ if and only if $t' \models \phi$ for all t' such that $t \prec_p t'$, etc. A formula ϕ is *valid in a model* if $t \models \phi$ for all $t \in T$ and it is *valid in a frame* if it is valid in every model based on it.

We say that a property of frames is *characterized by* an axiom if the axiom is valid in any frame that satisfies the property and, conversely, if a frame does not satisfy the property then the axiom is not valid in it. The next two propositions give the axioms that characterize the principle of minimum revision of prediction.

Proposition 7 *Property* (MR_a) *is characterized by the following axiom*

$$(A_{MR_a}) \qquad F_p(\phi_3 \wedge P(\phi_2 \wedge P_p\phi)) \to P\phi \vee F\phi \vee F_p(\phi_3 \wedge P_p(\phi_2 \wedge P_p\phi)).$$

Proof. Fix an arbitrary[6] frame that satisfies (MR_a). Let t_1, ϕ, ϕ_2 and ϕ_3 be such that $t_1 \models F_p(\phi_3 \wedge P(\phi_2 \wedge P_p\phi))$. Then there exist t_2, t_3 and t_4 such that $t_1 \prec_p t_3$, $t_3 \models \phi_3$, $t_2 \prec t_3$, $t_2 \models \phi_2$, $t_4 \prec_p t_2$ and $t_4 \models \phi$. By (MR_a) either $t_4 \prec t_1$, in which case $t_1 \models P\phi$, or $t_1 \prec t_4$, in which case $t_1 \models F\phi$, or $t_2 \prec_p t_3$, in which case $t_1 \models F_p(\phi_3 \wedge P_p(\phi_2 \wedge P_p\phi))$. Conversely, fix a frame that does not satisfy (MR_a). Then there exist t_1, t_2, t_3 and t_4 such that $t_1 \prec_p t_3$, $t_2 \prec t_3$, $t_4 \prec_p t_2$, $t_4 \not\prec t_1$, $t_1 \not\prec t_4$ and $t_2 \not\prec_p t_3$. Let q, q_2 and q_3 be sentence

[6]Note that the proofs of Propositions 7-10 do not make use of the properties that define a branching-time frame, that is, the characterization holds for every triple $\langle T, \prec, \prec_p \rangle$ where $\prec$ is *any* binary relation on T and $\prec_p$ is a subrelation of $\prec$.

letters and construct a model where $||q|| = \{t_4\}$ (recall that $||\phi||$ denotes the truth set of ϕ), $||q_2|| = \{t_2\}$ and $||q_3|| = \{t_3\}$. Then $t_1 \models F_p(q_3 \wedge P(q_2 \wedge P_p q))$. Since $t_4 \not\prec t_1$, $t_1 \not\models Pq$. Since $t_1 \not\prec t_4$, $t_1 \not\models Fq$. The only t at which $q_2 \wedge P_p q$ is true is t_2. Hence, since $t_2 \not\prec_p t_3$, $t_3 \not\models P_p(q_2 \wedge P_p q)$. Thus, since q_3 is true only at t_3, there is no t at which $q_3 \wedge P_p(q_2 \wedge P_p q)$ is true. Thus $t_1 \not\models F_p(q_3 \wedge P_p(q_2 \wedge P_p q))$. ■

Proposition 8 *Property* (MR_b) *is characterized by the following axiom*

$$(A_{MR_b}) \qquad F_p P(P_p\phi \wedge F_p\psi) \rightarrow P\phi \vee F\phi \vee F_p\psi.$$

Proof. Fix an arbitrary frame that satisfies (MR_b). Let t_1, ϕ and ψ be such that $t_1 \models F_p P(P_p\phi \wedge F_p\psi)$. Then there exist t_2, t_3, t_4 and t_5 such that $t_1 \prec_p t_3$, $t_2 \prec t_3$, $t_5 \prec_p t_2$, $t_5 \models \phi$, $t_2 \prec_p t_4$ and $t_4 \models \psi$. By (MR_b) either $t_5 \prec t_1$, in which case $t_1 \models P\phi$, or $t_1 \prec t_5$, in which case $t_1 \models F\phi$, or $t_1 \prec_p t_4$, in which case $t_1 \models F_p\psi$. Conversely, fix a frame that does not satisfy (MR_b). Then there exist t_1, t_2, t_3, t_4 and t_5 such that $t_1 \prec_p t_3$, $t_2 \prec t_3$, $t_5 \prec_p t_2$, $t_2 \prec_p t_4$, $t_5 \not\prec t_1$, $t_1 \not\prec t_5$ and $t_1 \not\prec_p t_4$. Let q, and r be sentence letters and construct a model where $||q|| = \{t_5\}$ and $||r|| = \{t_4\}$. Then $t_1 \models F_p P(P_p q \wedge F_p r)$. Since $t_5 \not\prec t_1$, $t_1 \not\models Pq$. Since $t_1 \not\prec t_5$, $t_1 \not\models Fq$. Finally, since $t_1 \not\prec_p t_4$, $t_1 \not\models F_p r$. ■

Thus in prediction frames the conjunction of axioms (A_{MR_a}) and (A_{MR_b}) characterizes the principle of minimum revision of prediction (MR) (cf. Corollary 4).

We now turn to the notion of consistency of prediction expressed by (CP). The following two propositions provide a characterization which corresponds to the informal verbal interpretation given in the previous section (for an alternative characterization see Bonanno, 2001, Lemma 2.4).

Proposition 9 *Property* (CP_1) *is characterized by the following axiom:*

$$(A_{CP_1}) \qquad F_p P\phi \rightarrow \phi \vee P\phi \vee F_p\phi.$$

Proof. Fix an arbitrary frame that satisfies (CP_1) and a model based on it. Suppose that, for some instant t_1 and formula ϕ, $t_1 \models F_p P\phi$. Then there exist $t_2, t_3 \in T$ such that $t_1 \prec_p t_3$, $t_2 \prec t_3$ and $t_2 \models \phi$. If $t_1 = t_2$, then $t_1 \models \phi$. If $t_2 \prec t_1$, then $t_1 \models P\phi$. Suppose, therefore, that $t_1 \neq t_2$ and $t_2 \not\prec t_1$. Then, by (CP_1), $t_1 \prec_p t_2$ and hence $t_1 \models F_p\phi$. Thus in all three cases $t_1 \models \phi \vee P\phi \vee F_p\phi$. Conversely, fix a frame that does not satisfy (CP_1). Then there exist $t_1, t_2, t_3 \in T$ such that $t_1 \prec_p t_3$, $t_2 \prec t_3$, $t_1 \neq t_2$, $t_2 \not\prec t_1$ and $t_1 \not\prec_p t_2$. Let q be a sentence letter and consider a model where $\|q\| = \{t_2\}$. Then all of the following are true at t_1: $F_p Pq$, $\neg q$ (because $t_1 \neq t_2$), $\neg Pq$ (because $t_2 \not\prec t_1$) and $\neg F_p q$ (because $t_1 \not\prec_p t_2$). ■

Proposition 10 *Property* (CP_2) *is characterized by the following axiom:*

$$(A_{CP_2}) \qquad F_p(\psi \wedge P\phi) \rightarrow \phi \vee P\phi \vee F_p(\psi \wedge P_p\phi).$$

Proof. Fix an arbitrary frame that satisfies (CP_2) and a model based on it. Suppose that, for some instant t_1 and formulas ϕ and ψ, $t_1 \models F_p(\psi \wedge P\phi)$. Then there exist $t_2, t_3 \in T$ such that $t_1 \prec_p t_3$, $t_3 \models \psi$, $t_2 \prec t_3$ and $t_2 \models \phi$. If $t_1 = t_2$, then $t_1 \models \phi$. If $t_2 \prec t_1$, then $t_1 \models P\phi$. Suppose, therefore, that $t_1 \neq t_2$ and $t_2 \not\prec t_1$. Then, by (CP_2), $t_2 \prec_p t_3$ so that $t_3 \models \psi \wedge P_p\phi$ and thus $t_1 \models F_p(\psi \wedge P_p\phi)$. Hence in all three cases we have that $t_1 \models \phi \vee P\phi \vee F_p(\psi \wedge P_p\phi)$. Conversely, fix a frame that does not satisfy (CP_2). Then there exist $t_1, t_2, t_3 \in T$ such that $t_1 \prec_p t_3$, $t_2 \prec t_3$, $t_1 \neq t_2$, $t_2 \not\prec t_1$ and $t_2 \not\prec_p t_3$. Let q and r be a sentence letters and consider a model where $||q|| = \{t_2\}$ and $\|r\| = \{t_3\}$. Then the following are true at t_1: $F_p(r \wedge Pq)$, $\neg q$ (because $t_1 \neq t_2$) and $\neg Pq$ (because $t_2 \not\prec t_1$). Furthermore, since $t_2 \not\prec_p t_3$ and q is true only at t_2, $t_3 \not\models P_pq$. Hence, since r is true only at t_3, $r \wedge P_pq$ is false everywhere. Thus $t_1 \models \neg F_p(r \wedge P_pq)$. ■

The conjunction of (A_{CP_1}) and (A_{CP_2}) thus provide a characterization of (CP) in prediction frames (cf. Proposition 5). The next proposition gives a *single* axiom which characterizes (CP) in prediction frames.

Proposition 11 *In prediction frames, property* (CP) *is characterized by the following axiom:*

$$(A_{CP}) \qquad F_p(\psi \wedge P\phi) \rightarrow \phi \vee P\phi \vee F_p(\phi \wedge F_p\psi).$$

Proof. First we show that in arbitrary frames (that is, frames $\langle T, \prec, \prec_p\rangle$ where $\prec$ is *any* binary relation on T and $\prec_p$ is a subrelation of $\prec$) axiom (A_{CP}) characterizes the following property: $\forall t_1, t_2, t_3 \in T$,
(CP'') if $t_1 \prec_p t_3$ and $t_2 \prec t_3$ then either $t_1 = t_2$ or $t_2 \prec t_1$ or ($t_1 \prec_p t_2$ and $t_2 \prec_p t_3$)

Fix an arbitrary frame that satisfies (CP''), a model based on it, an instant t_1 and formulas ϕ and ψ. Suppose that $t_1 \models F_p(\psi \wedge P\phi)$. Then there exist t_2 and t_3 such that $t_1 \prec_p t_3$, $t_3 \models \psi$, $t_2 \prec t_3$ and $t_2 \models \phi$. By (CP'') either $t_1 = t_2$, in which case $t_1 \models \phi$, or $t_2 \prec t_1$, in which case $t_1 \models P\phi$, or $t_1 \prec_p t_2$ and $t_2 \prec_p t_3$, in which case $t_1 \models F_p(\phi \wedge F_p\psi)$. Conversely, fix a frame that does not satisfy (CP''). Then there exist t_1, t_2 and t_3 such that $t_1 \prec_p t_3$, $t_2 \prec t_3$, $t_1 \neq t_2$, $t_2 \not\prec t_1$ and either $t_1 \not\prec_p t_2$ or $t_2 \not\prec_p t_3$. Let q and r be sentence letters and construct a model where $||q|| = \{t_2\}$ and $||r|| = \{t_3\}$. Since $t_1 \prec_p t_3$ and $t_2 \prec t_3$, $t_1 \models F_p(r \wedge Pq)$. Since $t_1 \neq t_2$, $t_1 \models \neg q$ and, since $t_2 \not\prec t_1$, $t_1 \models \neg Pq$. Furthermore, $(q \wedge F_pr)$ is either false everywhere, if $t_2 \not\prec_p t_3$, or it is true exactly at t_2, if $t_2 \prec_p t_3$, in which case, by our supposition, $t_1 \not\prec_p t_2$; thus in either case $t_1 \models \neg F_p(q \wedge F_pr)$. Next we prove that, in prediction frames, (CP'') is equivalent to (CP). $(CP) \Longrightarrow (CP'')$: Let $t_1 \prec_p t_3$ and $t_2 \prec t_3$. Since $\prec_p$ is a subrelation of $\prec$, $t_1 \prec t_3$. Hence by backward linearity of $\prec$, either $t_1 = t_2$

or $t_2 \prec t_1$ or $t_1 \prec t_2$. If $t_1 \prec t_2$ then by (CP) $t_1 \prec_p t_2$ and $t_2 \prec_p t_3$. $(CP'') \Longrightarrow (CP)$: Let $t_1 \prec_p t_3$, $t_1 \prec t_2$ and $t_2 \prec t_3$. Since $t_1 \prec t_2$, by asymmetry of $\prec$, $t_1 \neq t_2$ and $t_2 \not\prec t_1$. Thus, by (CP''), $t_1 \prec_p t_2$ and $t_2 \prec_p t_3$. ■

In Bonanno (2001) a system of logic is given which is sound and complete with respect to the class of prediction frames that satisfy (CP). This system uses axioms which are different from (A_{CP_1}), (A_{CP_2}) and (A_{CP}). Similar soundness and completeness results can be obtained for the systems discussed above. For example, one can show that the system obtained by adding to the logic of branching time axioms (A_p), (A_{MR_a}) and (A_{MR_b}) is sound and complete with respect to the class of prediction frames that satisfy (MR), etc.

4 Further properties of prediction

In this section we discuss other properties of prediction which might be appropriate in some contexts. Because of space limitations the proofs are omitted.

In some applications it may make sense to require that, for every instant t, the predicted future of t be non-empty, unless t is a terminal instant (i.e. it has no $\prec$-successors).[7] Semantically this property can be expressed as the requirement that $\prec_p$ be serial whenever $\prec$ is serial. Such a requirement rules out "agnosticism" in that it demands that some prediction be made, whenever possible. Note, however, that it is not at all a strong requirement, since the trivial prediction that "anything can happen" is consistent with this requirement: it corresponds to the case where $\prec_p = \prec$. The following proposition gives the axiom that characterizes this property.

Proposition 12 *The axiom $G_p\phi \wedge F\phi \rightarrow F_p\phi$ characterizes the property that $\prec_p$ is serial whenever $\prec$ is serial, that is, $\forall t \in T$, if $t \prec t_1$ for some t_1, then $t \prec_p t_2$ for some t_2.*

Another possible requirement is that predicted instants belong to a unique history, in the sense that the predicted future of any instant t consist of points on the same branch out of t. This requirement is captured by the following property:

[7]For an application of this principle to game theory see Bonanno (1998).

$$\text{if } t_1 \prec_p t_2 \text{ and } t_1 \prec_p t_3, \text{ then either } t_2 = t_3, \text{ or } t_2 \prec t_3 \text{ or } t_3 \prec t_2$$

whose characterizing axiom is

$$F_p\phi \wedge F_p\psi \rightarrow F_p(\phi \wedge \psi) \vee F_p(\phi \wedge F\psi) \vee F_p(F\phi \wedge \psi).$$

Finally, in some contexts a natural property to require is transitivity of $\prec_p$, whose characterizing axiom is $G_p\phi \rightarrow G_pG_p\phi$. Transitivity of $\prec_p$ can be viewed as capturing a principle of coherence of belief close in spirit to van Fraassen's Reflection Principle (van Fraassen, 1984): if I predict that at some future time I will visit France and I anticipate that, once I am on French soil, I will predict visiting Italy, then it seems that I ought to predict now that I will visit Italy at some future time.[8]

5 Conclusion

When we make a prediction we select, among the several possible future descriptions of the world, those that appear to us to be more plausible, thereby expressing our belief about what the future will be like. As time progresses, we revise our earlier predictions in the light of the actual unfolding of events. The principle of minimum revision of belief, applied to this context, says that, if there are no surprises, that is, if what we observe confirms our earlier predictions, then we should maintain all the past predictions that are still possible and not add any new ones. We provided a semantic and syntactic characterization of this principle. We also discussed and characterized a notion of consistency of predictions as well as possible further properties that one might want to impose on the notion of prediction.

[8] Transitivity of $\prec_p$ is a natural property whenever prediction is interpreted qualitatively as possibility or plausibility of future states. On the other hand, if prediction is interpreted probabilistically in terms of *likely* future states then transitivity of $\prec_p$ would not be a natural property: I may consider it likely today that I will reach age 75 and, if I indeed reached age 75, I would then consider it likely that I would live to be 80 and yet I might not consider it likely *today* that I will reach age of 80.

References

[1] Battigalli, Pierpaolo and Giacomo Bonanno, The logic of belief persistence, *Economics and Philosophy*, 1997, 13: 39-59.

[2] van Benthem, Johan, *The logic of time*, Kluwer Academic Publishers, 1991.

[3] Bonanno, Giacomo, Branching time, perfect information games and backward induction, Working Paper, University of California Davis, 1998. Forthcoming in *Games and Economic Behavior.*

[4] Bonanno, Giacomo, Prediction in branching time logic, *Mathematical Logic Quarterly*, 2001, 47.

[5] Burgess, John, Basic tense logic, in: D. Gabbay and F. Guenthner (eds.), *Handbook of philosophical logic*, Vol. II, 1984, 89-133, D. Reidel Publishing Company.

[6] van Fraassen, Bas, Belief and the will, *Journal of Philosophy*, 1984, 81: 235-256.

[7] Gärdenfors, Peter, *Knowledge in flux*, MIT Press, 1988.

[8] Halpern, Joseph, The relationship between knowledge, belief and certainty, *Annals of Mathematics and Artificial Intelligence*, 1991, 4: 301-322.

[9] Harman, Gilbert, *Change in view: principles of reasoning*, MIT Press, 1986.

[10] Hintikka, Jaakko, *Knowledge and belief,* Cornell University Press, 1962.

[11] van der Hoek, Wiebe, Systems for knowledge and belief, *Journal of Logic and Computation*, 1993, 3: 173-195.

[12] Lenzen, Wolfgang, Recent work in epistemic logic, *Acta Philosophica Fennica*, 1978, 30: 1-220.