Merging Frameworks for Interaction: DEL and ETL

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1 Introduction

Many logical systems today describe intelligent interacting agents over time. Frameworks include Interpreted Systems (IS, Fagin et al. [5]), Epistemic-Temporal Logic (ETL, Parikh & Ramanujam [13]), STIT (Belnap et al. [4]), Process Algebra and Game Semantics (Abramsky [1]). This variety is an asset, as different modeling tools can be fine-tuned to specific applications. But it may also be an obstacle, when barriers between paradigms and schools go up.

This paper takes a closer look at one particular interface, between two systems that both address the dynamics of knowledge and information flow in multi-agent systems. One is IS/ETL (IS and ETL are basically the same up to model transformations, cf. [11]), which uses linear or branching time models with added epistemic structure induced by agents' different capabilities for observing events. These models provide a Grand Stage where histories of some process unfold constrained by a protocol, and a matching epistemic-temporal language describes what happens. The other framework is Dynamic Epistemic Logic (DEL, [6, 3]) which describes interactive processes in terms of epistemic event models which may occur inside modalities of the language. Temporal evolution is then computed from some initial epistemic model through a process of successive 'product updates'. It has long been unclear how to best compare IS/ETL and DEL. [6, 19, 20] have investigated various aspects, but in this paper, we strengthen the interface to a considerable extent.

We first show how to transform DEL protocols into classes of ETL models, leading to a simple language translation from dynamic modalities to temporal operators. Next, we prove a new *representation theorem* characterizing the largest class of ETL models corresponding to DEL protocols in terms of notions of Perfect Recall, No Miracles, and Bisimulation Invariance. These describe the sort of idealized agent presupposed in standard DEL. Next, we consider further assumptions on agents, and introduce a new technique of modal *correspondence analysis* relating special properties of DEL protocols to corresponding ETL-style properties. Finally, we how the DEL ETL analogy suggests new issues of *completeness*. Our new contribution is an axiomatization for the dynamic logic of public announcements constrained by protocols, which has been an open problem for some years, as it does not fit the usual 'reduction axiom' format of DEL.

Once again, we are not reducing one framework to another. We show rather how ETL and DEL lead to interesting new issues when merged as accounts of intelligent agents.

2 Relating the Two Frameworks

Epistemic Temporal Logic: We start with the basics of ETL. Let Σ be any set and $\mathcal{A}$ a finite set of agents. Elements of Σ are called **events**, and elements of the set of finite strings Σ^* **histories**. For any two sets X and Y, XY is the set of sequences consisting of an object in X followed by one in Y. Given $h \in \Sigma^*$, the **length** of h ($\mathsf{len}(h)$) is the number of events in h. Given $h, h' \in \Sigma^*$, we write $h \preceq h'$ if h is a prefix of h'. Let λ be the empty string. For a set of histories $\mathcal{H} \subseteq \Sigma^*$, $\mathsf{FinPre}_{-\lambda}(\mathcal{H}) = \{h \mid h \text{ is nonempty and } \exists h' \in \mathcal{H} \text{ such that } h \preceq h'\}$. Given an event $e \in \Sigma$, we write $h \prec_e h'$ if $h' = he$.

Definition 2.1 (ETL Structures) Let Σ be any set of events. A **protocol** is a set $\mathcal{H} \subseteq \Sigma^*$ with $\mathsf{FinPre}_{-\lambda}(\mathcal{H}) \subseteq \mathcal{H}$. An **ETL frame** is a tuple $\langle \Sigma, \mathcal{H}, \{\sim_i\}_{i\in\mathcal{A}}\rangle$ with Σ a (finite or infinite) set of events, $\mathcal{H}$ a protocol, and for each $i \in \mathcal{A}$, $\sim_i$ is a binary relation[1] on $\mathcal{H}$. An **ETL model** is a tuple $\langle \Sigma, \mathcal{H}, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle$ where V is a valuation $V : \mathsf{At} \to 2^{\mathcal{H}}$ and $\langle \Sigma, \mathcal{H}, \{\sim_i\}_{i\in\mathcal{A}}\rangle$ an ETL frame. ◁

[1] Although we will not do so here, typically it is assumed that each $\sim_i$ is an equivalence relation.

We write $\sim^*$ for the reflexive transitive closure of the union of the $\sim_i$ relations. A protocol $\mathcal{H}$ can be seen as a forest of trees. The intended interpretation is that each $h \in H$ represents a certain point in time in the evolution of an interactive situation (such as a game or conversation), with h' such that $h \prec_e h'$ representing the point in time after e has happened in h. As usual, the relations $\sim_i$ represent the uncertainty of the agents about how the situation has evolved.

Different modal languages describe these structures (see [9]), with 'branching' or 'linear' variants. Here we give just the bare necessities. Let At be a countable set of atomic propositions. Formulas are interpreted at histories $h \in \mathcal{H}$. The basic propositional modal language $\mathcal{L}_{EL}$ has epistemic operators for each agent (K_i), and extended with temporal operators for each event $e \in \Sigma$ (N_e) it becomes the larger language $\mathcal{L}_{ETL}$. Truth is defined as usual: see [5] and [9] for details. We only recall the definition of the knowledge and the temporal operators:

- $h \models K_i\phi$ iff for each $h' \in \mathcal{H}$, if $h \sim_i h'$ then $h' \models \phi$
- $h \models N_e\phi$ iff there exists $h' \in \mathcal{H}$ such that $h \prec_e h'$ and $h' \models \phi$

It is often natural to extend the language $\mathcal{L}_{ETL}$ with group knowledge operators (eg., common or distributed knowledge) and more expressive temporal operators (eg., arbitrary future or past modalities). This may lead to high complexity of the validity problem (cf. [8, 20] and Section 5).

Dynamic Epistemic Logic: An alternative account of interactive dynamics was elaborated by [6, 3, 16, 21] and others. From an initial epistemic model, temporal structure evolves as needed.

Definition 2.2 (DEL Structures) An **epistemic model** is a tuple $M = \langle W, \{R_i\}_{i\in\mathcal{A}}, V\rangle$ where $R_i \subseteq W \times W$ and V is a valuation function ($V : \mathsf{At} \to 2^W$). The set W is the domain of M, denoted $\mathcal{D}(M)$. An **event model** E is a tuple $\langle S, \longrightarrow_i, \mathsf{pre}\rangle$, where S is a nonempty set of events, $\longrightarrow_i \subseteq S \times S$ and $\mathsf{pre} : S \to \mathcal{L}_{EL}$. The set S is called the domain of E, denoted $\mathcal{D}(E)$.

The **product update** $M \times E$ of an epistemic model M with an event model E is the epistemic model (W', R'_i, V') such that $W' = \{(w, e) \mid w \in W, e \in S$ and $M, w \models \mathsf{pre}(e)\}$, $(w, e)R_i(w', e')$ iff wR_iw' in M and $e \longrightarrow_i e'$ in E, and $V'((s, e)) = V(s)$. ◁

The language $\mathcal{L}_{DEL}$ extends $\mathcal{L}_{EL}$ with operators $\langle E, e\rangle$ for each pair of event models E and event e in the domain of E. Truth for $\mathcal{L}_{DEL}$ is defined as usual. We only define the typical DEL modalities: $M, w \models \langle E, e\rangle\phi$ iff $M, w \models \mathsf{pre}(e)$ and $M \times E, (w, e) \models \phi$.

From DEL Protocols to ETL Models: Our key observation is that by repeatedly updating an epistemic model with event models, the machinery of DEL in effect creates ETL models. To make this precise, let a **DEL protocol** be a set $\mathcal{E}$ of finite sequences of pointed event models closed under the initial segment relation (cf. Definition 2.1)[2]. For simplicity, for each DEL protocol $\mathcal{E}$, we let the domains of each event model in $\mathcal{E}$ be disjoint. Let $\mathcal{D}(\mathcal{E})$ be their union.

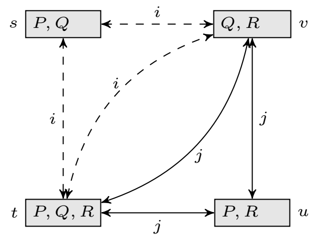
Definition 2.3 (DEL Generated ETL Models) Let M be an epistemic model, and $\mathcal{E}$ a DEL protocol. The ETL model generated by M and $\mathcal{E}$, $\mathsf{Forest}(M, \mathcal{E})$, represents all possible evolutions of the system obtained by updating M with sequences from $\mathcal{E}$. It is a disjoint union of models of the form $M \times E_1 \times \cdots E_n$ where $(E_1E_2 \ldots E_n) \in \mathcal{E}$. More formally, $\mathsf{Forest}(M, \mathcal{E}) = \langle \Sigma, \mathcal{H}, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle$ with $\Sigma = \{s \mid s \in W\} \cup \{e \mid e \in \mathcal{D}(\mathcal{E})\}$ and $\mathcal{H} \subseteq \mathcal{D}(M)\mathcal{D}(E)^*$. The uncertainty relations are copied from the models $M \times E_1 \times \cdots \times E_n$, and the temporal relations ($\prec_e$ for each $e \in \mathcal{D}(\mathcal{E})$) are the initial segment relation as above. If $\mathcal{E}$ is a protocol, we set $\mathbb{F}(\mathcal{E}) = \{\mathsf{Forest}(M, \mathcal{E}) \mid$ for all epistemic models $M\}$. ◁

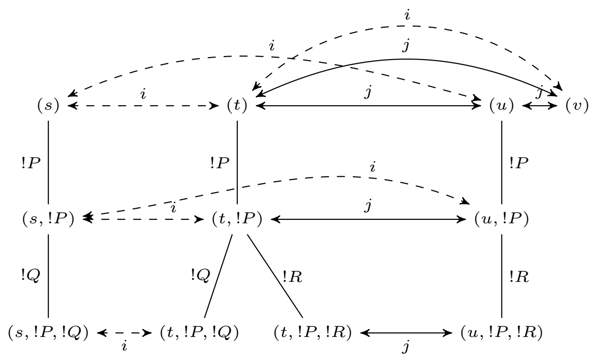
Because $\mathcal{E}$ is closed under prefixes, so is the domain of $\mathsf{Forest}(M, \mathcal{E})$. Hence, Definition 2.3 indeed describes an ETL model. We illustrate this construction with an example.

Example: In *public announcement logic* (PAL [14]), each event model denotes an announcement $!A$ of some true formula A. Thus it consists of a single point with one reflexive arrow for each agent and the precondition is A. The corresponding operators $\langle !A\rangle\phi$ mean: "after publicly announcing A, ϕ is true". The product update model resulting from an initial model M and a public announcement model E is simply the submodel of M consisting of all states where P is true. Now, suppose that $\mathcal{E} = \{(!P), (!P, !Q), (!P, !R)\}$ and consider the figure below. The initial epistemic model M is displayed on the left and the generated ETL model $\mathsf{Forest}(M, \mathcal{E})$ is on the right. Note that in this example $\mathsf{Forest}(M, \mathcal{E}), (t) \models R \wedge \neg\langle !R\rangle\top$. Thus even though a formula is true, it may not be "announcable" due to the underlying protocol. This raises issues to be discussed in Section 5.

Matching our model transformation, there is also a translation between languages. Think of the DEL op-

[2] The *preconditions of DEL* also encode protocol information (cf. [16]). We do not pursue this line here.





erators $\langle E, e\rangle$ as labelled temporal operators. This defines a translation $(\cdot)^{\#} : \mathcal{L}_{DEL} \to \mathcal{L}_{ETL}$ as follows: $(\cdot)^{\#}$ commutes over boolean connectives, is the identity map on the set of propositional variables, and[3] $(\langle E, e\rangle\phi)^{\#} = N_{E,e}\phi^{\#}$. This translation preserves truth in the following sense. Let $\mathcal{DEL}$ be the protocol of all finite sequences of event models. Let M be an epistemic model, $w \in \mathcal{D}(M)$, and hence $(w) \in \mathsf{Forest}(M, \mathcal{DEL})$.

Proposition 2.4 For any formula $\phi \in \mathcal{L}_{DEL}$, $M, w \models \phi$ iff $\mathsf{Forest}(M, \mathcal{DEL}), (w) \models \phi^{\#}$.

Proposition 2.4 explains a common intuition about linking DEL to ETL. But there is more to come!

3 Representation results

Not all ETL models can be generated by a DEL protocol. Indeed, such generated ETL models have a number of special properties. In this section we study precisely which properties these are.

First we note that standard DEL events do not change ground facts. Let $T = \langle \Sigma, \mathcal{H}, \{\sim_i\}_{i\in\mathcal{A}}\rangle$ be an ETL frame. We say T satisfies **propositional stability** iff for all $h \in \mathcal{H}$, $e \in \Sigma$ with $he \in \mathcal{H}$, $h \models p$ iff $he \models p$. Our second property reflects the fact that in product update, uncertainty does not cross between M and $M \times E$. We say T satisfies **synchronicity** iff for all $h, h' \in \mathcal{H}$, if $h \sim_i h'$, then $\mathsf{len}(h) = \mathsf{len}(h')$. The further properties come from the definition of product update and vary depending on one's class of DEL protocols. We start by characterizing the ETL models resulting from consecutive updates with one single event model.

Definition 3.1 (Epistemic Bisimilar) A relation $\sim$ over histories in $\mathcal{H}$ is an *epistemic bisimulation* when for all h and h', if $h \sim h'$, then (1) h and h' satisfy the same atomic propositions, (2) for every h'' with $h \sim_i h''$, there is a h''' with $h' \sim_i h'''$; and vice versa. If some epistemic bisimulation connects h and h', we say that h and h' are *epistemically bisimilar*. ◁

Definition 3.2 (ETL Properties) Let $T = \langle \Sigma, H, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle$ be an ETL model. T satisfies:

- *Perfect Recall* iff for all $h, h' \in \mathcal{H}$, $e, e' \in \Sigma$ with $he, h'e' \in \mathcal{H}$, if $he \sim_i h'e'$, then $h \sim_i h'$
- *No Miracles* iff for all $h, h' \in \mathcal{H}$, $e, e' \in \Sigma$ with $he, h'e' \in \mathcal{H}$, if there are $h'', h''' \in \mathcal{H}$ with $h''e, h'''e' \in \mathcal{H}$ such that $h''e \sim_i h'''e'$ and $h \sim_i h'$, then $he \sim_i h'e'$.
- *Bisimulation Invariance* iff for all epistemically bisimilar $h, h' \in \mathcal{H}$, if $he \in \mathcal{H}$ then $h'e \in \mathcal{H}$. ◁

Let E be a fixed event model and $\mathcal{E}_E$ be the protocol that consists of all finite sequences of the repetition of the single event model E. That is $\mathcal{E}_E = \{h \mid h \in \{\mathcal{D}(E)\}^* - \{\lambda\}\}$.

Proposition 3.3 (van Benthem [16]) An ETL model T is of the form $\mathsf{Forest}(M, \mathcal{E}_E)$ for some M and E iff T satisfies propositional stability, synchronicity, perfect recall, no miracles and bisimulation invariance.

But there are many further DEL protocols $\mathcal{E}$ of interest[4]. E.g., to model 'conversation', let $\mathbb{F}(PAL)$ consist of all models $\mathsf{Forest}(M, \mathcal{E})$ with $\mathcal{E}$ involving just public announcements.

Proposition 3.4 (PAL-generated models) An ETL model $\langle \Sigma, \mathcal{H}, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle$ is in $\mathbb{F}(PAL)$ iff it is synchronous, propositionally stable, satisfies the minimal properties of Theorem 3.6, and:

- for all $h, h', he, h'e \in \mathcal{H}$, if $h \sim_i h'$ then $he \sim_i h'e$ (all events are reflexive)
- for all $h, h' \in \mathcal{H}$, if $he \sim_i h'e'$, then $e = e'$ (no two different events are connected).

[3] We also have versions with more standard temporal operators N_e which we leave to the full paper.

[4] Van Benthem & Liu [19] suggest that iterating one large event model involving suitable preconditions can 'mimic' ETL style evolution for more complex protocols. We do not pursue this claim here.

But our first main new result in this paper is a characterization of the class of all DEL generated models.

Definition 3.5 Let $T = \langle \Sigma, H, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle$ be an ETL model. T satisfies:

- *Local No Miracles* iff for all $h_1, h_2, h, h' \in \mathcal{H}$, $e, e' \in \Sigma$ with $h_1e, h_2'e' \in \mathcal{H}$, if $h_1e \sim_i h_2e'$ and $h \sim_i h'$ and $h \sim^* h'$ then $he \sim_i h'e'$ (if $he, h'e' \in \mathcal{H}$)
- *Local Bisimulation Invariance* iff for all $h, h' \in \mathcal{H}$, if $h \sim^* h'$ and h and h' are epistemically bisimilar, and $he \in \mathcal{H}$, then $h'e \in \mathcal{H}$ $\triangleleft$

Theorem 3.6 Let $\mathcal{DEL}$ be the class of *all* DEL protocols. A model is in $\mathbb{F}(\mathcal{DEL})$ iff it satisfies synchronicity, perfect recall, local uniform no miracles, and local bisimulation invariance.

This Theorem identifies the *minimal properties* that any DEL generated model must satisfy, and thus it describes exactly what type of agent is presupposed in the DEL framework. The proof generalises the one in van Benthem & Liu [19], which is an immediate special case. The proof of the characterization of PAL (Proposition 3.4) is also a simple variant. The reader is referred to [18] for details.

Remark 3.7 Given our interest in epistemic temporal languages, one might ask for variants of Theorem 3.6 with models characterized only up to some *epistemic-temporal bisimulation.* (But eg., Perfect Recall is not preserved this way). Cf. again [18].

4 Correspondence Results

Our representation theorems suggest a more general correspondence theory relating natural properties of ETL frames with axioms in suitable modal languages. Our method of generating ETL models with DEL protocols gives us a new way of describing ETL frames – we can look for classes of frames that are generated by particular types of DEL protocols.

Definition 4.1 (Frame characterization) A *formula* ϕ **characterizes an ETL frame property** P iff all and only frames in which ϕ is valid have property P. A *property* P^{DEL} *of DEL protocols* characterizes a ETL frame property P iff all and only *DEL* generated frames with P are generated by a protocol with P^{DEL}. $\triangleleft$

$\mathcal{L}_{ETL}$ is only one of many languages for reasoning about DEL generated ETL models, and there are many other temporal and epistemic operators of interest in reasoning about these models. Formulas of the form $F\phi$ say that "ϕ is true sometime in the future", $N_{e^*}\phi$ says that "ϕ is true after a finite sequence of e events" and $C\phi$ says that "ϕ is common knowledge". Formally, let $T = \langle \Sigma, \mathcal{H}, \{\sim_i\}_{i\in\mathcal{A}}, V\rangle$ be an ETL model. If $e \in \Sigma$ and n a natural number, then e^n is the sequence of $ee\cdots e$ of length n. We can also add "backwards-looking" operators with formulas $Y_e\phi$ meaning that ϕ was true before event e happened (and e happened just before).

- $h \models F\phi$ iff there exists $h' \in \mathcal{H}$, $h \preceq h'$ and $h' \models \phi$.
- $h \models N_{e^*}\phi$ iff there exists $h' \in \mathcal{H}$ where $h' = he^n$ for some $n \geq 0$ and $h' \models \phi$
- $h \models C\phi$ iff for each $h' \in \mathcal{H}$, if $h \sim^* h'$ then $h' \models \phi$
- $h \models Y_e\phi$ iff there exists $h' \in \mathcal{H}$ such that $h' \prec_e h$ and $h' \models \phi$

The second main contribution of this paper is a set of correspondences showing that a more general theory is feasible here[5]. The Tables below summarize a number of results; some known, some new. The first two rows correlate ETL frame properties with their characterizing formulas in the sense of the first item in the Definition 4.1. The first and third rows correlate frame properties with protocols as in the second item from Definition 4.1. For more precise formulations and all proofs, we refer to Appendix A. Here we just discuss what the Tables say.

(1) **Reflexivity**

Frame Property	if $h \prec_e h'$ and $h'' \prec_e h'''$ and $h \sim_i h''$, then $h' \sim_i h'''$
Axiom Scheme	$N_eK_i\phi \to K_iN_e^\circ\phi$
DEL Protocol	$e \longrightarrow_i e$

(2) **Commutativity**

Frame Property	if $h \prec_e h'$, $h' \sim_i h_1$, then there is an h_2 with $h \sim_i h_2$ and $h_2 \prec_e h_1$
Axiom Scheme	$N_eL_i\phi \to L_iN_e\phi$
DEL Protocol	$e \longrightarrow_i f$ only if $e = f$

(3) **Functionality**

Frame Property	if $h \prec_e h'$ and $h \prec_e h''$, then $h' = h''$
Axiom Scheme	$N_e\phi \to N_e^\circ\phi$
DEL Protocol	all protocols

[5] [15] discusses some related correspondence issues but with out our new connection to DEL protocols.

(4) **Perfect Observability**

Frame Property	if $h \prec_e h'$, $h \prec_f h''$, $h' \sim_i h''$, then $e = f$.
Axiom Scheme	$N_e^\circ K_i \neg N_{f^-} \top$
DEL Protocol	$e \longrightarrow_i f$ only if $e = f$

(5) **Perfect Recall**

Frame Property	if $h \prec_e h'$ and $h'' \prec_e h'''$ and $h' \sim_i h'''$, then $h \sim_i h''$
Axiom Scheme	$N_e L_i N_{f^-} \phi \to L_i \phi$
DEL Protocol	updates introduce only relations present in the epistemic model

(6) **No Miracles**

Frame Property	If $h \prec_e h'$ and $h_1 \prec_f h_1'$ and $h' \sim_i h_1'$, and if $h_2 \prec_e h_2'$ and $h_3 \prec_f h_3'$ and $h_2 \sim_i h_3$, and $h_2 \sim^* h$, then $h' \sim_i h_1'$.
Axiom Scheme	$\langle C \rangle N_e L_i N_{f^-} \top \to (N_e K_i \phi \to K_i N_f^\circ \phi)$ ($\langle C \rangle = \neg C \neg$)
DEL Protocol	

In the above table, N_e° is $\neg N_e \neg$, L_i is $\neg K_i \neg$ and N_{f^-} is the converse of N_f. Properties (1) and (2) distinguish PAL protocols. So there is a relation between their frame axioms and the axioms of public announcement logic. And indeed, if in the PAL reduction axiom $[!A]K_i\phi \leftrightarrow (A \to K_i[!A]\phi)$, we replace the public announcement $!A$ with an arbitrary event label, and its precondition A with the sentence $N_e\top$ (the precondition for an occurrence of e in the ETL-model) this becomes: $N_e^\circ K_i\phi \leftrightarrow (N_e\top \to K_i N_e^\circ \phi)$. In the presence of functionality (3), the two implications in this equivalence are provably equivalent to the axioms in (1) and (2).

Item (4) highlights the fact that "perfect observability" – if an event takes place, you know that no other event takes place – cannot be characterized within the class of *all ETL* frames with the "forward-looking" operators only: we need "backwards-looking" operators as well. Also perfect recall (5) and no miracles (6) cannot be characterized by forward-looking formulas – the latter needs common knowledge as well. As all DEL generated models satisfy these properties, there are no particular protocols that distinguish them. Still, perfect recall captures exactly that having sR_is' in the original model is a necessary condition for having $(s,e)R_i(s',e')$ in the new model.

5 Axiomatization and Completeness

Representation theorems as in Section 3, or correspondence results as in Section 4, are two ways of describing the DEL-ETL interface. But there is also the familiar approach of completeness theorems. Here we discuss a number of languages and axiomatization results.

Here are two natural classes of DEL induced ETL models. The first is $\mathbb{F}(\mathcal{E})$: all ETL models $\mathsf{Forest}(M,\mathcal{E})$ generated from a *specific* DEL protocol $\mathcal{E}$. An example is $\mathbb{F}(\mathcal{DEL})$, the class of all ETL structures generated by the 'full protocol' of all possible sequences of DEL events. But also of interest are the ETL models coming from a fixed *set of DEL protocols* $\mathbf{X}$. We define $\mathbb{F}\mathbf{X} = \{\mathsf{Forest}(M,\mathcal{E}) \mid M$ an epistemic model and $\mathcal{E} \in \mathbf{X}\}$. E.g., if $\mathbf{X}_{DEL} = \{\mathcal{E} \mid \mathcal{E}$ is a DEL protocol$\}$, $\mathbb{F}\mathbf{X}_{DEL}$ contains all ETL structures generated by some DEL protocol.

The move to special sets of protocols is non-trivial. For instance, consider again the crucial 'reduction axiom' $[!A]K_i\phi \leftrightarrow (A \to K_i[!A]\phi)$ of public announcement logic (PAL). This drives the compositional analysis of epistemic postconditions, and in the end, it reduces every dynamic-epistemic formula to an equivalent epistemic one in $\mathcal{L}_{EL}$. But this key axiom does have a presupposition: the assertion A, if true, is always *available for announcement.* If we no longer assume this — as is natural in conversational scenarios — the usual DEL completeness results are in jeopardy! We return to this observation below, but first, we review known results for full protocols.

5.1 Logics of Specific Protocols

'Full protocols' have been the norm in DEL so far. Let $\mathcal{PAL}$ be the protocol of all possible public announcements (i.e., all finite sequences of formulas from $\mathcal{L}_{EL}$). The usual axiomatization PAL of public announcement logic works for this class. Similarly, the logic of $\mathbb{F}(\mathcal{DEL})$ is the standard axiomatization of DEL [3, 21]). But with extended languages the situation becomes more diverse. It is argued in [16] that in the full PAL protocol, there is a sequence of public announcements that can change *implicit knowledge* of ground facts into *common knowledge.* In other words, for ground formulas ϕ, $D\phi \to FC\phi$ is valid in $\mathbb{F}(\mathcal{PAL})$, where $D\phi$ is distributed knowledge of ϕ.

This table summarizes what we know about complete logics for such extended languages (F.A. stands for 'Finite Axiomatizable' and $EPDL$ stands for *epistemic propositional dynamic logic.* See [21] for details.):

Language	$\mathbb{F}(\mathcal{PAL})$	$\mathbb{F}(\mathcal{DEL})$
K_i, N_e	F.A. [14]	F.A. [3]
K_i, N_e, C	F.A. [3]	F.A. [3]
$EPDL$, N_e	F.A. [21]	F.A. [21]
K_i, N_e, F	F.A. [2]	Open
K_i, N_e, N_{e^*}	Not F.A. [10]	Open
K_i, N_{e^*}	Open	Open
K_i, N_e, C, N_{e^*}	Not F.A. [10]	Open

Miller & Moss [10] show that $\mathbb{F}^{\infty}_{\mathcal{E}_0} = \{\mathsf{Forest}(M, \mathcal{E}_0) \mid M \text{ infinite }\}$ where $\mathcal{E}_0 = \{L_i\top\}^*$ is not even axiomatizable for languages that contain knowledge modalities and arbitrary future modalities. There are many further questions here (cf. [20]): we refer to the full version of the paper.

5.2 Logics of Protocol Sets

Our main new observation is about real scenarios for conversations. Unlike 'full protocols', these restrict the available assertions. Logics for their generated ETL models have not been explored yet.

First consider $\mathbb{F}\mathbf{X}_{PAL} = \{\mathsf{Forest}(M, \mathcal{E}) \mid M$ an epistemic model, $\mathcal{E}$ a PAL protocol$\}$ and the language $\mathcal{L}_{ETL}$. This is the space of all possible 'conversation scenarios'. Example 2 already showed that the standard axiomatization of PAL will not work here. Truth of A is no longer equivalent to $\langle !A\rangle\top$, the availability of A for assertion in our scenario. This invalidates the usual axioms of PAL – and we must redo the job. Our third main result of this paper shows that we can!

Definition 5.1 (TPAL Logic) The *logic of conversation* is the set **TPAL**:

PC Any axiomatization of propositional calculus

$\mathbf{K}_i$ $K_i(\phi \to \psi) \to (K_i\phi \to K_i\psi)$

R1 $\langle !A\rangle p \leftrightarrow \langle !A\rangle\top \wedge p$

R2 $\langle !A\rangle\neg\phi \leftrightarrow \langle !A\rangle\top \wedge \neg\langle !A\rangle\phi$

R3 $\langle !A\rangle(\phi \wedge \psi) \leftrightarrow \langle !A\rangle\phi \wedge \langle !A\rangle\psi$

R4 $\langle !A\rangle K_i\phi \leftrightarrow \langle !A\rangle\top \wedge K_i(A \to \langle !A\rangle\phi)$

A1 $\langle !A\rangle(\phi \to \psi) \to (\langle !A\rangle\phi \to \langle !A\rangle\psi)$

A2 $\langle !A\rangle\top \to A$

which is closed under modus ponens and necessitation for K_i and $[!A]$. ◁

These axioms illustrate the mixture of factual and procedural truth which drives conversations. A few remarks are in order. Axiom $R1$ illustrates that, in an arbitrary PAL protocol, truth of A does not guarantee that A *can* be announced. Second, axiom $R4$ hides a subtlety. One would expect this 'procedure-oriented' axiom: $\langle !A\rangle K_i\phi \leftrightarrow \langle !A\rangle\top \wedge K_i(\langle !A\rangle\top \to \langle !A\rangle\phi)$. The point is, however, that in our setting, announcements are *uniform actions*: if A can be announced at some history h and agent i knows A, then A can be announced in all i-equivalent histories. Indeed, the corresponding theorem $\langle !A\rangle\top \to K_i(A \to \langle !A\rangle\top)$ is derivable in **TPAL** (Lemma B.7).

Theorem 5.2 *TPAL* is sound and complete with respect to the class $\mathbb{F}\mathbf{X}_{PAL}$.

The proof is no longer a routine exercise in dynamic to epistemic reduction; and so we put the main steps of the proof in Appendix B. The situation is still more interesting with language extensions. Consider, sub-protocols of the $\mathbf{X}_{PAL}$. In a simple dialogue, we could identify the content of a statement of ϕ by an agent i with a public announcement that $K_i\phi$ – agents can only say what they know to be true. Protocols built from such announcements have special properties. We mention one observation from [6]: the information present in the initial model – called "combined knowledge" in [6] and "the communicative core" in [16] – will not grow (or diminish). With an operator I expressing this notion, our protocol logic would encode this as the validity of $I\phi \to GI\phi$.

Sets of DEL protocols also formalize further phenomena (cf. [12, 16]). Consider, for example, the classic "coordinated attack" problem ([5]) where no new facts can become common knowledge. Now, let $\mathbf{X}'$ be the set of DEL protocols containing sequences of event models with two events, one with precondition ϕ, the other with the trivial precondition. The sender's accessibility relation connects the events, that of the receiver is the identity relation. We can prove a parallel observation: $C\phi \leftrightarrow GC\phi$ is valid in $\mathbb{F}\mathbf{X}'$.

But the general logic of DEL protocol sets seems wide open. It is likely that results of Halpern, van der Meyden and Vardi [7] are relevant here. We still have to do the math!

6 Conclusions

Epistemic-temporal logic and dynamic-epistemic logic are two major and interestingly different ways of describing knowledge-based interaction over time. We have shown how the two can be linked in three ways: using representation theorems, modal correspondence analysis, and new sorts of axiomatic completeness theorems for epistemic-temporal model classes generated by DEL protocols. Our results suggest a more systematic 'logic of protocols' using ideas from DEL to add fine structure to ETL.

As for extensions, one should increase the descriptive scope of our analysis to deal with changing beliefs over time. This seems quite feasible, using doxastic-temporal logics and recent versions of DEL for belief change [17]. The other challenge that we see is using DEL, with its explicit account of model construction inside the logic, as an intermediate between ETL-style frameworks which describe properties of states and histories inside given models, and paradigms like process algebra or game semantics, with their explicit construction of dynamic processes.

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A Correspondence Proofs

Proposition A.1 (1) Let $\mathcal{F}$ be the frames satisfying:

$$\text{If } s \prec_e t \text{ and } s' \prec_e t' \text{ and } s \sim_i s', \text{ then } t \sim_i t'$$

Then $\mathcal{F}$ is exactly the class characterized by the following axiom: $N_e K_i \phi \to K_i \neg N_e \neg \phi$. Also, the DEL-generated frames with this property are exactly those generated by reflexive models.

Proof. The correspondence between frame property and axiom can be done with standard methods, and is straightforward.

We show that $\mathcal{F} = \{\mathsf{Forest}(\mathcal{M}, \mathcal{E}) \mid \mathcal{E}$ contains reflexive models only $\}$.

Let F be the frame of a model $\mathsf{Forest}(\mathcal{M}, \mathcal{E})$, for some reflexive $\mathcal{E}$. Suppose $s \sim_i s'$, $s \prec_e t$ and $s' \prec_e t'$. Then, by reflexivity and the definition of product update, $se \sim_i te$.

For the other direction, assume that F is a DEL-generated frame that satisfies the property. Consider the construction of the "canonical" protocol in the proof of Proposition 3.6 (see [18] for details), but change it slightly and define the accessibility relations $e \longrightarrow_i e'$ iff for all sequences se and $s'e'$ it holds that if $s \sim_i s'$ then $se \sim_i se'$. The proof that F is generated by this protocol works just the same, and it is easy to see that now the protocol must contain only reflexive events. QED

Proposition A.2 (2) and (4) The class of frames that satisfy: if $s \prec_e t$ and $t \sim_i t'$, then there is an s' with $s \sim_i s'$ and $s' \prec_e t'$ is characterized by the axiom

$$N_e L_i \phi \to L_i N_e \phi$$

The DEL-generated frames satisfying this property are exactly those generated by event models with: if $e \longrightarrow_i f$, then $e = f$.

Proof. The correspondence between commutativity and its modal axiom is well-known.

For the DEL-correspondence, suppose F is the frame of a model $\mathsf{Forest}(\mathcal{M}, \mathcal{E})$, for some $\mathcal{E}$ built with event models with the stated property. Suppose $se \sim_i te'$. Then, from the definition of product update, we know that $e \longrightarrow_i e'$ and $s \sim_i t$. By assumption, $e = e'$, and so $t \prec_e te'$.

For the other direction, consider the protocol that generates F that we constructed in the proof of proposition 3.6. Now, suppose that $e \longrightarrow_i e'$ in that model. By construction, that means that there must be se and te' in F such that $se \sim_i te'$. With our frame property, there must be an s' such that $s'e$ is in the model, and $s \sim_i s'$ and $s'e = te'$. But that means that $e = e'$.

As commutativity and perfect observability coincide on DEL frames, (4) is a corollary. QED

Properties (4), (5) cannot be expressed in the "forward-looking" language only:

Proposition A.3 The properties of *perfectly observable events*, *perfect recall* and *uniform no miracles* cannot be characterized in the forward-looking language

Proof. To prove this, we provide pairs of frames that validate the same sentences, one verifying and the other falsifying the relevant frame property. (We can see that the frames validate the same sentences by finding a total relation $\sim$ between the states of the frame such that if $s \sim s'$, then the generated subframe of s is isomorphic to the generated subframe of s' in the second frame.)

For *perfect observability*, compare the frame $s_0 \prec_e s_1 \sim_i t_1$ and $t_0 \prec_f t_1$ (with $e \neq f$) that falsifies perfect observability, with a frame that has $s_0 \prec_e s_1 \sim_i t_1$ and $t'_0 \prec_f t'_1$.

For *perfect recall*, we can use the same example.

For *uniform no miracles*, we can again use the same example with some added structure: both both models, add states $u_0 \prec_e u_1$ and $u_0 \sim_i v_0 \prec_f v_1$. QED

Definition A.4 (Generalized Update) A function U that takes Kripke models and event models to a new Kripke model is an *Update Function* iff the new model as as its domain all pairs (s, e) such that $s \models \mathsf{pre}(e)$; i.e. the new model has as the same domain as $M \times E$, but the exact nature of the accessibility relations remains undetermined. ◁

We can now talk about $\mathsf{Forest}(M, \mathcal{E}, U)$ as the forest generated by updating M along the lines of $\mathcal{E}$ as prescribed with U, and talk about properties of update functions characterizing frame properties in much the same way as in Definition 4.1. This abstract setting is related to the correspondence analyses for belief revision in [17].

Proposition A.5 (5) Update functions U such that if $se \sim_i s'e'$, then $s \sim_i s'$ generate exactly the models that satisfy perfect recall.

Proof. The "soundness" part is fairly straightforward – just check if the update functions generate the right kind of models, as in Theorem 3.6.

For the other direction, suppose U lacks the property. Then there is a model M and event model E with states s and s' in M and e and e' in E such that $se \sim_i s'e'$ with $s \not\sim_i s'$. But then the protocol starting with E, applied to M, lacks perfect recall. QED

B Completeness of $TPAL$

We give the details of the completeness of $TPAL$ discussed in Section 5. To make this section self-contained we first recall the definitions of the intended class of models and the language.

Definition B.1 (TPAL Language) Let At be a set of propositional variables (either finite or infinite) and $\mathcal{A}$ a (finite) set of agents. The **basic temporal public announcement language** is generated by the following grammar:

$$p \mid \neg\phi \mid \phi \wedge \psi \mid K_i\phi \mid \langle !\phi\rangle\psi$$

where $p \in \mathsf{At}$ and $i \in \mathcal{A}$. Let $\mathcal{L}_{TPAL}$ be the set of all formulas generated by this grammar. We use standard abbreviations for all further connectives, and for the modal operators $\langle i\rangle$ and $[!\phi]$. ◁

Definition B.2 (PAL Structures) Given a Kripke model $\mathcal{M} = \langle W, R_i, V\rangle$ and $\phi \in \mathcal{L}_{TPAL}$, the model $\mathcal{M} \times E_\phi = \langle W^{!\phi}, R_i^{!\phi}, V^{!\phi}$ where

- $W^{!\phi} = \{(w, \phi) \mid w \in W \text{ and } \mathcal{M}, w \models \phi\}$
- for each $(w, \phi), (v, \phi) \in W^{!\phi}$, $(w, \phi)R_i^{!\phi}(v, \phi)$ iff wR_iv
- for each $p \in \mathsf{At}$, $V^{!\phi}(p) = \{(w, \phi) \mid w \in V(p)\}$

We may also denote this model $\mathcal{M}^{!\phi}$. ◁

Given a sequence of formulas $\sigma := \phi_1\phi_2 \cdots \phi_n$ of formulas from $\mathcal{L}_{TPAL}$ and a Kripke model $\mathcal{M}$, we write

$\mathcal{M} \times E_\sigma$ for the model $(\cdots(\mathcal{M} \times E_{\phi_1}) \times E_{\phi_2}) \cdots \times E_{\phi_n}$. We denote this model $\langle W^\sigma, R^\sigma, V^\sigma \rangle$. The states W^σ of $\mathcal{M} \times E_\sigma$ are sequences starting with a state from $\mathcal{M}$ followed by σ.

Definition B.3 (TPAL Structures) A *TPAL*-**protocol** is a set $\mathcal{E}$ of finite sequences of formulas from $\mathcal{L}_{TPAL}$. For each sequence $\sigma \in \mathcal{E}$ where $\sigma = \phi_1\phi_2 \ldots \phi_n$ and Kripke model $\mathcal{M}$, $\mathsf{Forest}(\mathcal{M}, \mathcal{E})$ is the ETL-model $\langle \mathcal{H}, \sim_i, V \rangle$ where

- $\mathcal{H} = \{h \mid h$ is a state from $\mathcal{M} \times E_\sigma$ for some $\sigma \in \mathcal{E}\}$
- For each $h, h' \in \mathcal{H}$, $h \sim_i h'$ iff $hR_i^\sigma h'$ where $h = w\sigma$ and $h' = v\sigma$ for some $\sigma \in \mathcal{E}$.
- For each $p \in \mathsf{At}$ and $h \in \mathcal{H}$, $h \in V(p)$ iff $V^\sigma(p)$ where $h = w\sigma$ and $h' = v\sigma$ for some $\sigma \in \mathcal{E}$

$\mathsf{Forest}(TPAL)$ consists of all models $\mathsf{Forest}(\mathcal{M}, \mathcal{E})$ for some Kripke model $\mathcal{M}$ and protocol $\mathcal{E}$. ◁

Given a model $\mathsf{Forest}(\mathcal{M}, \mathcal{E}) = \langle \mathcal{H}, \sim_i, V \rangle$ truth of formulas $\phi \in \mathcal{L}_{TPAL}$ is defined as in Section 4. The atomic propositional variables and boolean connectives are as usual. We recall the definition of the modal operators: let $h \in \mathcal{H}$ and $t \in \mathbb{N}$,

- $h \models K_i\phi$ iff for each $h' \in \mathcal{H}$, if $h \sim_i h'$ then $h' \models \phi$
- $h \models \langle !\psi \rangle \phi$ iff $h\psi \in \mathcal{H}$ and $h\psi \models \phi$

Definition B.4 (TPAL Logic) The *TPAL-logic* is the set **TPAL** of all instances of

PC Any axiomatization of propositional calculus

$\mathbf{K}_i$ $K_i(\phi \to \psi) \to (K_i\phi \to K_i\psi)$

R1 $\langle !A \rangle p \leftrightarrow \langle !A \rangle \top \wedge p$

R2 $\langle !A \rangle \neg\phi \leftrightarrow \langle !A \rangle \top \wedge \neg\langle !A \rangle \phi$

R3 $\langle !A \rangle (\phi \wedge \psi) \leftrightarrow \langle !A \rangle \phi \wedge \langle !A \rangle \psi$

R4 $\langle !A \rangle K_i\phi \leftrightarrow \langle !A \rangle \top \wedge K_i(A \to \langle !A \rangle \phi)$

A1 $\langle !A \rangle (\phi \to \psi) \to (\langle !A \rangle \phi \to \langle !A \rangle \psi)$

A2 $\langle !A \rangle \top \to A$

which is closed under modus ponens and necessitation for K_i and $[!A]$. ◁

Consistency, satisfiability and validity are defined entirely as usual.

Theorem B.5 TPAL is sound and strongly complete with respect to the class $\mathsf{Forest}(\mathsf{PAL})$.

The proof is in Henkin-style. We show that any consistent set of formulas is satisfiable in some model. By a Lindenbaum Lemma, every consistent set of formulas can be extended to a maximally consistent set. We now describe how to construct the canonical model. To simplify notation we write $\mathcal{L}$ for $\mathcal{L}_{TPAL}$.

Let $\mathbb{M} = \{\Gamma \mid \Gamma$ is a maximally consistent subset of $\mathcal{L}_{TPAL}\}$. Consider the set $\mathbb{M} \cdot \mathcal{L}^*$ of sequences of maximally consistent sets followed by sequences of formulas from $\mathcal{L}$. We write σ_j for the σ_j for the jth element of the sequence (thus $\sigma_0 \in \mathbb{M}$ and for each $j > 0$, $\sigma_j \in \mathcal{L}$).

Now, certain sequences $\sigma \in \mathbb{M} \cdot \mathcal{L}^*$ are legal as a possible sequence of public announcements. We attach a maximally consistent set to each legal finite sequence σ. To this end, we define sets $H_n \subseteq \mathbb{M} \cdot \mathcal{L}^*$ of legal sequences of length n and maps from H_n to $\mathbb{M}$ ($\lambda_n : H_n \to \mathbb{M}$) as follows:

- For $n = 0$, define $H_0 = \mathbb{M}$ and for each $\Gamma \in H_0$, $\lambda(\Gamma) = \Gamma$
- Let $H_{n+1} = \{\sigma A \mid \sigma \in H_n$ and $\langle !A \rangle \top \in \lambda(\sigma)\}$. Let $\sigma = \sigma' A \in H_{n+1}$ and define $\lambda_{n+1}(\sigma) = \{\phi \mid \langle !A \rangle \phi \in \lambda_n(\sigma')\}$.

We first show that each map λ_n is well-defined.

Lemma B.6 For each $n \geq 0$, for each $\sigma \in H_n$, $\lambda_n(\sigma)$ is a maximally consistent set.

Proof. Induction on n. The case $n = 0$ is by definition. Suppose that the statement holds for H_n and λ_n. Suppose $\sigma \in H_{n+1}$ with $\sigma = \sigma' A$. By the induction hypothesis, $\lambda_n(\sigma')$ is a maximally consistent set. Furthermore, by the construction of H_{n+1}, $\langle !A \rangle \top \in \lambda_n(\sigma)$. Therefore, $\lambda_{n+1}(\sigma) \neq \emptyset$. Let $\phi \in \mathcal{L}$. Since $\lambda_n(\sigma')$ is a maximally consistent set either $\langle !A \rangle \phi \in \lambda_n(\sigma')$ or $\neg\langle !A \rangle \phi \in \lambda_n(\sigma')$. If $\langle !A \rangle \phi \in \lambda_n(\sigma')$, by construction $\phi \in \lambda_{n+1}(\sigma)$. If $\neg\langle !A \rangle \phi \in \lambda_n(\sigma')$, by axiom $R2$, $\langle !A \rangle \neg\phi \in \lambda_n(\sigma')$. Hence, by construction $\neg\phi \in \lambda_{n+1}(\sigma)$. Thus for all $\phi \in \mathcal{L}$, either $\phi \in \lambda_{n+1}(\sigma)$ or $\neg\phi \in \lambda_{n+1}(\sigma)$.

To show $\lambda_{n+1}(\sigma)$ is consistent we argue by contradiction. Suppose there are $\phi_1, \ldots, \phi_m \in \lambda_{n+1}(\sigma)$ such that $\vdash \wedge_{j=1}^m \phi_j \to \bot$. Using standard modal reasoning, $\vdash \wedge_{j=1}^{m-1} \langle !A \rangle \phi_j \to \langle !A \rangle \neg\phi_m$. Since for each $j = 1, \ldots, m$, $\langle !A \rangle \phi_j \in \lambda_n(\sigma')$, we have $\langle !A \rangle \neg\phi_m \in \lambda_n(\sigma')$. Using axiom $R2$ (recall $\langle !A \rangle \top \in \lambda_n(\sigma')$), $\neg\langle !A \rangle \in \lambda_n(\sigma')$. This contradicts the fact that $\lambda_n(\sigma')$ is consistent. QED

Let $H_{can} = \cup_{n\geq 0} H_n$. Define $\lambda : H \to \mathbb{M}$ as follows: for each $\sigma \in H$, $\lambda(\sigma) = \lambda_n(\sigma)$ where n is the length of σ (denote $\mathsf{len}(\sigma)$). The canonical model $T_{can} = (H_{can}, \{\approx_i\}_{i\in\mathcal{A}}, V_{can})$ is defined as follows:

- $H_{can} = \cup_{n\geq 0} H_n$.
- $\approx_i$ is the smallest relation satisfying the following closure conditions:
  - If $\sigma, \tau \in H_{can}$ are sequences of length one (i.e., $\sigma = (\Gamma)$ and $\tau = (\Delta)$ where $\Gamma, \Delta \in \mathbb{M}$) then $\sigma \approx_i \tau$ iff$_{\text{def}}$ $\{\phi \mid K_i\phi \in \lambda(\sigma)\} \subseteq \lambda(\tau)$
  - If $\sigma, \tau \in H_{can}$ are of the form $\sigma = \sigma'\phi$ and $\tau = \tau'\phi$, then $\sigma \approx_i \tau$ iff$_{\text{def}}$ $\sigma' \approx_i \tau'$.
- for each $p \in \mathsf{At}$, $V_{can}(p) = \{\sigma \mid p \in \lambda(\sigma)\}$

Lemma B.7 The formula $\langle !A\rangle\top \to K_i(A \to \langle !A\rangle\top)$ is derivable in **TPAL**.

Proof. Using standard modal reasoning we can derive $\langle !A\rangle\top \to \langle !A\rangle K_i\top$ using the fact that $K_i\top$ is derivable and $A1$. As an instance of $R4$, we can derive $\langle !A\rangle K_i\top \leftrightarrow \langle !A\rangle\top \wedge K_i(A \to \langle !A\rangle\top)$. Thus, $\mathbf{TPAL} \vdash \langle !A\rangle\top \to \langle !A\rangle\top \wedge K_i(A \to \langle !A\rangle\top)$. By propositional reasoning, $\mathbf{TPAL} \vdash \langle !A\rangle\top \to K_i(A \to \langle !A\rangle\top)$. QED

Lemma B.8 (Truth Lemma) For each $\phi \in \mathcal{L}$ and $\sigma \in H_{can}$, $\phi \in \lambda(\sigma)$ iff $T_{can}, \sigma \models \phi$.

Proof. The proof is by induction on the structure of ϕ. As usual, the boolean connectives and the base case are easy. We only show the modal case:

Suppose ϕ is of the form $K_i\psi$ and the statement holds for ψ. Suppose $\sigma = \Gamma A_1 A_2 \cdots A_n$ for some $n \geq 0$ and $K_i\psi \in \lambda(\sigma)$. Suppose there is some $\tau \in H_{can}$ such that $\sigma \approx_i \tau$. By construction of the canonical model this means $\tau = \Delta A_1 A_2 \cdots A_n$ with $\Gamma \approx_i \Delta$ (and each subsequence of the same length are equivalent, but this is not needed). Since $K_i\psi \in \lambda(\Gamma A_1 \cdots A_n)$, we have $\langle !A_n\rangle K_i\psi \in \lambda(\Gamma A_1 \cdots A_{n-1})$. Hence, using $R4$, $K_i(A_n \to \langle !A_n\rangle\psi) \in \lambda(\Gamma A_1 \cdots A_{n-1})$. Hence, $\langle !A_{n-1}\rangle K_i(A_n \to \langle !A_n\rangle\psi) \in \lambda(\Gamma A_1 \cdots A_{n-2})$ and so $K_i(A_{n-1} \to \langle !A_{n-1}\rangle(A_n \to \langle !A_n\rangle(\psi))) \in \lambda(\Gamma A_1 \cdots A_{n-2})$. Continuing in this manner, we have

$$K_i(A_1 \to \langle !A_1\rangle(A_2 \to \langle !A_2\rangle(\cdots(A_n \to \langle !A_n\rangle\psi)))) \in \Gamma$$

Since $\Gamma \approx_i \Delta$, by construction of the canonical model,

$$A_1 \to \langle !A_1\rangle(A_2 \to \langle !A_2\rangle(\cdots(A_n \to \langle !A_n\rangle\psi))) \in \Delta \quad (*)$$

Furthermore, since $\tau = \Delta A_1 \cdots A_n \in H_{can}$, $\langle !A_1\rangle\top \in \Delta$ and for $k = 2, \ldots, n$, $\langle !A_k\rangle\top \in \lambda(\Delta \cdots A_{k-1})$. Using $A2$, this implies $A_1 \in \Delta$ and $k = 2, \ldots, n$, $A_k \in \lambda(\Delta \cdots A_{k-1})$. Hence, by $(*)$ and this fact, $\langle !A_1\rangle(A_2 \to \langle !A_2\rangle(\cdots(A_n \to \langle !A_n\rangle\psi))) \in \Delta$. Therefore, $(A_2 \to \langle !A_2\rangle(\cdots(A_n \to \langle !A_n\rangle\psi))) \in \lambda(\Delta A_1)$. Continuing in this manner, we see that $\psi \in \lambda(\tau)$. By the induction hypothesis, $T_{can}, \tau \models \psi$. Since τ is arbitrary and $\sigma \approx_i \tau$, we have $T_{can}, \sigma \models K_i\psi$.

For the other direction, suppose that $K_i\psi \notin \lambda(\sigma)$. For simplicity, we assume $\sigma = \Gamma A$. This makes the argument easier to follow, but can easily be generalized as above. By construction of σ, $\langle !A\rangle\top \in \Gamma$ and so by $A4$, we have $K_i(A \to \langle !A\rangle\psi) \notin \Gamma$. If we can find a maximally consistent set Δ such that $\Gamma \approx_i \Delta$, $\langle !A\rangle\top \in \Delta$ and $\langle !A\rangle\psi \notin \Delta$, then we are done. In this case, $\Gamma A \approx_i \Delta A$ and $\psi \notin \lambda(\Delta A)$. Thus by the induction hypothesis, $T_{can}, \Delta A \not\models \psi$ and so $T_{can}, \Gamma A \not\models K_i\psi$. Let $\Delta' = \{\chi \mid K_i\chi \in \Gamma\} \cup \{\neg(A \to \langle !A\rangle\phi)\}$. We claim that Δ' is consistent. Suppose not. Then there are $\chi_1, \ldots, \chi_m$ such that for each $j = 1, \ldots, m$, $K_i\chi_j \in \Gamma$ and $\mathbf{TPAL} \vdash \bigwedge_{j=1,\ldots,m} \chi_i \to (A \to \langle !A\rangle\phi)$. Using standard modal reasoning, $\mathbf{TPAL} \vdash \bigwedge_{j=1,\ldots,m} K_i\chi_j \to K_i(A \to \langle !A\rangle\phi)$. Thus, since for each $j = 1, \ldots, m$, $K_i\chi_j \in \Gamma$, we have $K_i(A \to \langle !A\rangle\phi) \in \Gamma$. As Γ is a maximally consistent set, this contradicts the assumption that $K_i(A \to \langle !A\rangle\psi) \notin \Gamma$. Thus Δ' is consistent and, by Lindenbaum's Lemma, can be extended to a maximally consistent set Δ with $\Gamma \approx_i \Delta$. Note that since $\langle !A\rangle\top \in \Gamma$, by Lemma B.7, $K_i(A \to \langle !A\rangle\top) \in \Gamma$. Therefore, $A \to \langle !A\rangle\top \in \Delta$. Since $\neg(A \to \langle !A\rangle\phi) \in \Delta$, we have $A \in \Delta$ and $\langle !A\rangle\phi \notin \Delta$. Thus, $\langle !A\rangle\top \in \Delta$.

Suppose ϕ is of the forms $\langle !A\rangle\psi$ and the statement holds for ψ. Suppose that $\langle !A\rangle\psi \in \lambda(\sigma)$. This implies $\langle !A\rangle\top \in \lambda(\sigma)$ (this follows since for any ψ, by standard modal reasoning $\mathbf{TPAL} \vdash \langle !A\rangle\psi \to \langle !A\rangle\top$). Therefore, $\sigma A \in H_{can}$ and by definition, $\psi \in \lambda(\sigma A)$. Hence, by the induction hypothesis, $T_{can}, \sigma A \models \psi$. Therefore, $T_{can}, \sigma \models \langle !A\rangle\psi$. For the other direction, suppose that $T_{can}, \sigma \models \langle !A\rangle\psi$. Then by definition of truth, $\sigma A \in H_{can}$ and $T_{can}, \sigma A \models \psi$. By the induction hypothesis, $\psi \in \lambda(\sigma A)$. Hence, by definition, $\langle !A\rangle\psi \in \lambda(\sigma)$. QED

The proof of Theorem B.5 now follows using standard arguments.