

# Knowledge at equilibrium<sup>1</sup>

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June 1995

## Abstract

In a game with rational expectations individuals refine their information with the information revealed by the acts of other individuals. We show that, at a Nash equilibrium of a game with rational expectations, the information of individuals is essentially symmetric, and their acts are common knowledge.

JEL classification number: D82.

Keywords: Nash equilibrium; rational expectations; common knowledge.

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<sup>1</sup> This text presents research results of the Belgian programme on Interuniversity Poles of attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by its authors. The Commission of the European Union provided additional support through the SPES grant CT91-5047 and the HMC grant ERBCHRXCT-940458.

We wish to thank P. Battigalli, F. Donzelli, F. Forges, L. Koutsougeras and A. Rustichini for their comments.

## 1. Introduction

Individuals may have different private information. When this is the case, the outcomes of their interaction reflect the information of all, and the observation of the outcomes may convey new information to each.

This is the motivation for the introduction of *rational expectations* in general equilibrium theory in Radner (1979): if the individuals understand the relationship between states of the world and prices, their choices must be optimal with respect to the information *at equilibrium*, their original, private information refined by the information revealed by prices.

Nash equilibrium profiles in a standard bayesian game in Harsanyi (1967-68) need not have this property. The strategies chosen by the individuals are optimal with respect to their original, private information, and the observation of the outcome of the play may well cause some individual to regret his choice. This is justified by the claim that a game is a complete description of the situation, so that, strictly speaking, there is never a moment in which regret matters. Still, one of the justifications for the use of Nash equilibrium makes reference to the steady state of a recurring interaction. We can think of sequences of individuals being drawn to play the same game. If there is no uncertainty about the payoffs and if the whole record of previously played actions is known to each individual when he is called to play, the steady state, if it exists, will be a Nash equilibrium of the stage game. The natural extension of this interpretation to situations of incomplete information requires a notion of rational expectations in bayesian games.

In this note we introduce a notion of rational expectations in bayesian games and we show that at a Nash equilibrium of a game with rational expectations the information of individuals is essentially symmetric: any differences in information do not affect equilibrium acts, and, equivalently, the acts of individuals are *common knowledge*.

Common knowledge of acts immediately implies consensus among individuals trying to guess the value of a random variable. Speculative behaviour, on the other hand, cannot be excluded at an equilibrium of the game with rational expectations. Indeed, an example shows that common knowledge of the acts need not imply common knowledge of speculation. This clarifies previous claims to no-speculation as the result of common knowledge of willingness to trade, as in Milgrom and Stokey (1982), or of

rational expectations, as in Tirole (1982), or even simply of Nash equilibrium, as in Geanakoplos (1989).

## 2. Games with rational expectations

A game with uncertainty is a collection

$$\mathbf{G}_{\mathbf{S}} = \{\mathbf{H}, \mathbf{S}, (\mathbf{A}^h, u^h) : h \in \mathbf{H}\}.$$

Individuals are  $h \in \mathbf{H}$ , a non-empty, finite set. States of the world are  $s \in \mathbf{S}$ , a non-empty, finite set. For an individual, an elementary act or an elementary strategy is  $a^h \in \mathbf{A}^h$ , and an act or a strategy is  $f^h \in \mathbf{F}^h = \{f^h : \mathbf{S} \rightarrow \mathbf{A}^h\}$ . Across individuals, a profile of elementary acts is  $a^{\mathbf{H}} \in \mathbf{A}^{\mathbf{H}}$ , where  $a^{\mathbf{H}} = (\dots, a^h, \dots)$ , and  $\mathbf{A}^{\mathbf{H}} = \times_{h \in \mathbf{H}} \mathbf{A}^h$ , and a profile of acts is  $f^{\mathbf{H}} \in \mathbf{F}^{\mathbf{H}} = \{f^{\mathbf{H}} : \mathbf{S} \rightarrow \mathbf{A}^{\mathbf{H}}\}$ .

The utility or payoff to the individual at  $f^{\mathbf{H}} \in \mathbf{F}^{\mathbf{H}}$  is  $u^h(f^{\mathbf{H}})$ , and his utility function is  $u^h : \mathbf{F}^{\mathbf{H}} \rightarrow \mathbf{R}$ .

At acts  $f^{\mathbf{H} \setminus \{h\}} \in \mathbf{F}^{\mathbf{H} \setminus \{h\}}$  by the complementary set of individuals, where  $f^{\mathbf{H} \setminus \{h\}} = (\dots, f^{h-1}, f^{h+1}, \dots)$ ,  $\mathbf{F}^{\mathbf{H} \setminus \{h\}} = \times_{h' \in \mathbf{H} \setminus \{h\}} \mathbf{F}^{h'}$ , and  $f^{\mathbf{H}} = (f^h, f^{\mathbf{H} \setminus \{h\}})$ , the optimization problem of the individual is to maximize  $u^h(f^h, f^{\mathbf{H} \setminus \{h\}})$  subject to  $f^h \in \mathbf{F}^h$ . The solution to the optimization problem is  $\varphi^h(f^{\mathbf{H} \setminus \{h\}}) \subseteq \mathbf{F}^h$ , which may be empty, when a solution does not exist, or not a singleton, when the solution is not unique. The choice or reaction correspondence is  $\varphi^h : \mathbf{F}^{\mathbf{H} \setminus \{h\}} \rightarrow \mathbf{F}^h$ .

A Nash equilibrium is a profile of acts,  $f^{\mathbf{H}^*} = (\dots, f^{h^*}, \dots)$ , such that  $f^{h^*} \in \varphi^h(f^{\mathbf{H} \setminus \{h\}^*})$ , for  $h \in \mathbf{H}$ .

The set of elementary acts does not vary across states of the world. More importantly, the set of acts, across states of the world, has a product structure.

In a game with uncertainty, individuals act with knowledge of the state of the world or, equivalently, with complete information.

A profile of private information is  $\mathcal{P}^{\mathbf{H}} = \{\dots, \mathcal{P}^h, \dots\}$ , where  $\mathcal{P}^h = \{\mathbf{P}^h(s) : s \in \mathbf{S}\}$  is a partition of the set of states of the world representing the private information of the individual.

For a game with uncertainty,  $\mathbf{G}_{\mathbf{S}}$ , and for a profile of private information,  $\mathcal{P}^{\mathbf{H}}$ , the game with uncertainty and private information<sup>1</sup> is

$$\mathbf{G}_{(\mathbf{S}, \mathcal{P}^{\mathbf{H}})} = \{\mathbf{H}, \mathbf{S}, (\mathbf{A}^h, \mathcal{P}^h, u^h) : h \in \mathbf{H}\}.$$

An act for an individual is  $f^h \in \mathbf{F}^h = \{f^h : \mathbf{S} \rightarrow \mathbf{A}^h, \text{ such that } f^h \text{ is measurable with respect to } \mathcal{P}^h\}$ .<sup>2</sup>

At a Nash equilibrium profile of acts,  $f^* = (\dots, f^{h*}, \dots)$ , with private information,  $f^{h*}$  is optimal with respect to the partition  $\mathcal{P}^h$ .

Individuals optimize ex-ante, prior to the resolution of uncertainty, and the utility function evaluates profiles of acts, profiles of elementary acts across states of the world. Under conditions which are well understood, Debreu (1959), the utility function of an individual is additively separable across states of the world or, equivalently, across profiles of elementary acts,  $u^h(f^{\mathbf{H}}) = \sum_{s \in \mathbf{S}} u_s^h(f^{\mathbf{H}}(s))$ . Under stronger conditions, Savage (1954), it has a state-independent expected utility representation,  $u^h(f^{\mathbf{H}}) = E^h v^h(f^{\mathbf{H}}(s))$ , where the index  $h$  on the expectation operator indicates that the probability measure under which expectations are computed is as much a characteristic which may vary across individuals as the cardinal utility index.

An ex-post formulation of games with uncertainty and private information is possible. For separable utility functions, the optimization problem of an individual at a state of the world  $s$  is

$$\max \sum_{s' \in \mathbf{P}(s)} u_{s'}^h(f^h(s), f^{\mathbf{H} \setminus \{h\}}(s')).$$

For non-separable utility functions, an ex-post formulation is possible but contrived. With this ex-post formulation of games with uncertainty, the solutions to the individual optimization problems coincide for  $s$  and  $s'$  with  $\mathbf{P}(s) = \mathbf{P}(s')$ . They yield unambiguously a solution to the ex-ante optimization problem, which, in particular,

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<sup>1</sup>Games with uncertainty and private information were formalized by Harsanyi (1967-68) and Aumann (1974).

<sup>2</sup>A function  $f : \mathbf{S} \rightarrow \mathbf{A}$  is measurable with respect to a partition  $\mathcal{P}$  if and only if  $\mathbf{P}(s) = \mathbf{P}(s') \Rightarrow f(s) = f(s')$ .

is measurable with respect to the information available to the individual.<sup>3</sup>

Information is symmetric if and only if, for some partition,  $\mathcal{P}$ , of the set of states of the world,  $\mathcal{P}^h = \mathcal{P}$ , for  $h \in \mathbf{H}$ : the information of individuals coincides. A game with uncertainty and symmetric information is

$$\mathbf{G}_{(\mathbf{S}, \mathcal{P})} = \{\mathbf{H}, \mathbf{S}, \mathcal{P}, (\mathbf{A}^h, u^h) : h \in \mathbf{H}\}.$$

Information is complete if and only if,  $\mathbf{P}^h(s) = \{s\}$ , for  $s \in \mathbf{S}$  and  $h \in \mathbf{H}$ . The game with uncertainty and complete information is, simply, the game with uncertainty.

At a Nash equilibrium,  $f^{\mathbf{H}^*}$ , of a game,  $\mathbf{G}_{(\mathbf{S}, \mathcal{P}^{\mathbf{H}})} = \{\mathbf{H}, \mathbf{S}, (\mathbf{A}^h, u^h, \mathcal{P}^h) : h \in \mathbf{H}\}$ , the information of individuals is essentially symmetric with respect to a partition,  $\mathcal{P}$ , of the set of states of the world, if and only if  $f^{\mathbf{H}^*}$  is a Nash equilibrium of the game  $\mathbf{G}_{(\mathbf{S}, \mathcal{P})} = \{\mathbf{H}, \mathbf{S}, \mathcal{P}, (\mathbf{A}^h, u^h) : h \in \mathbf{H}\}$  with symmetric information.

In a game with rational expectations, individuals refine their information with the information revealed by the elementary acts of other individuals at each state of the world.

For an individual, consider the feasible act correspondence defined by  $\psi^h(f^{\mathbf{H} \setminus \{h\}}) = \{f^h : \mathbf{S} \rightarrow \mathbf{A}^h, \text{ such that } f^h \text{ is measurable with respect to the partition } \mathcal{P}^h \vee_{h' \in \mathbf{H} \setminus \{h\}} \mathcal{P}^{h'}\}$ .<sup>4</sup> Equivalently, an act,  $f^h$ , is feasible for the individual if and only if, whenever

<sup>3</sup>An alternative ex-post formulation is for an individual to solve, in state  $s$ , the optimization problem

$$\max \sum_{s' \in \mathbf{P}(s)} u_{s'}^h(f^h(s), f^{\mathbf{H} \setminus \{h\}}(s)).$$

Evidently, solutions need not coincide if  $\mathbf{P}(s) = \mathbf{P}(s')$  and need not yield a solution to the ex-ante optimization problem subject to the measurability constraint.

<sup>4</sup>A partition,  $\mathcal{P}$ , is at least as coarse as another,  $\mathcal{P}'$ , or, equivalently,  $\mathcal{P}'$  is at least as fine as  $\mathcal{P}$ , if and only if  $\mathbf{P}'(s) = \mathbf{P}'(s') \Rightarrow \mathbf{P}(s) = \mathbf{P}(s')$ ; we write  $\mathcal{P}' \subseteq \mathcal{P}$ .

Let  $\{\mathcal{P}^k : k \in \mathbf{K}\}$  be a collection of partitions. The join is defined as the partition  $\overline{\mathcal{P}} = \vee_{k \in \mathbf{K}} \mathcal{P}^k$  such that  $\overline{\mathcal{P}}(s) = \overline{\mathcal{P}}(s')$  if and only if  $\mathbf{P}^k(s) = \mathbf{P}^k(s')$  for all  $k \in \mathbf{K}$ : it is the coarsest common refinement. The meet is defined as the partition  $\underline{\mathcal{P}} = \wedge_{k \in \mathbf{K}} \mathcal{P}^k$ , it is the finest common coarsening.

The partition,  $\mathcal{P}^f$ , induced by a function  $f$  is defined by  $\mathbf{P}^f(s) = \mathbf{P}^f(s')$  if and only if  $f(s) = f(s')$ : it is the coarsest partition with respect to which the function is measurable.

$f^h(s) \neq f^h(s')$ , either  $\mathbf{P}^h(s) \neq \mathbf{P}^h(s')$  - his private information allows the individual to distinguish states  $s$  and  $s'$  - or  $f^{h'}(s) \neq f^{h'}(s')$ , for some  $h' \in \mathbf{H} \setminus \{h\}$  - the elementary acts of some other individual,  $h'$ , allows, the individual to distinguish states  $s$  and  $s'$ .

A game with rational expectations is thus simply a generalized game<sup>5</sup>.

The optimization problem of the individual<sup>6</sup> is to maximize  $u^h(f^h, f^{\mathbf{H} \setminus \{h\}})$  subject to  $f^h \in \psi^h(f^{\mathbf{H} \setminus \{h\}})$ . The solution to the optimization problem is  $\varphi^h(f^{\mathbf{H} \setminus \{h\}}) \subseteq \psi^h(f^{\mathbf{H} \setminus \{h\}})$ , and the choice correspondence is  $\varphi^h : \mathbf{F}^{\mathbf{H} \setminus \{h\}} \rightarrow \mathbf{F}^h$ .

A profile of acts,  $f^{\mathbf{H}}$ , is feasible if and only if  $f^h \in \psi^h(f^{\mathbf{H} \setminus \{h\}})$ , for  $h \in \mathbf{H}$ .

A Nash equilibrium for the game with rational expectations is a feasible profile of acts,  $f^{\mathbf{H}^*}$ , such that  $f^{h^*} \in \varphi^h(f^{\mathbf{H} \setminus \{h\}})$ , for  $h \in \mathbf{H}$ .

The concept of a Nash equilibrium for a game with rational expectations abstracts from the strategic aspects of the revelation of information. This is in the spirit of Nash equilibrium, but may fall short of an interpretation of rational expectations as a reduced form of a process of information revelation and learning.

**Proposition 1.** At a Nash equilibrium for a game with rational expectations, the information of individuals is essentially symmetric with respect to the information partition  $\mathcal{P}^* = \vee_{h \in \mathbf{H}} \mathcal{P}^{f^{h^*}}$ .

**Proof:** Let  $f^{\mathbf{H}^*} = (\dots, f^{h^*}, \dots)$  be a profile of acts which is a Nash equilibrium with rational expectations. We shall show that  $f^{\mathbf{H}^*}$  is a Nash equilibrium for the game with symmetric information  $\mathbf{G}_{(\mathbf{S}, \mathcal{P}^*)}$ , where  $\mathcal{P}^* = \vee_{h \in \mathbf{H}} \mathcal{P}^{f^{h^*}}$ .

We argue in steps for each individual:

- i) By the definition of the partition induced by the function,  $f^{h^*}$  is measurable with respect to the partition  $\mathcal{P}^{f^{h^*}}$ , and hence also with respect to the finer partition  $\mathcal{P}^*$ .
- ii) By the definition of Nash equilibrium with rational expectations, the function  $f^{h^*}$  is measurable and optimal with respect to the partition  $\tilde{\mathcal{P}}^h = \mathcal{P}^h \vee_{h' \in \mathbf{H} \setminus \{h\}} \mathcal{P}^{f^{h'^*}}$ .

<sup>5</sup>Debreu (1952).

<sup>6</sup>In this generalized game  $\mathbf{F}^h = \{f : \mathbf{S} \rightarrow \mathbf{A}^h\}$ . Measurability restrictions are only imposed through the constraint correspondence  $\psi^h$ .

iii) Since the partition  $\mathcal{P}^{f^{h^*}}$  is the coarsest partition with respect to which the function  $f^{h^*}$  is measurable, the partition  $\tilde{\mathcal{P}}^h$  is at least as fine as the partition  $\mathcal{P}^{f^{h^*}}$ . It follows that the partition  $\tilde{\mathcal{P}}^h \vee_{h' \in \mathbf{H} \setminus \{h\}} \mathcal{P}^{f^{h'^*}}$  is at least as fine as the partition  $\mathcal{P}^{f^{h^*}} \vee_{h' \in \mathbf{H} \setminus \{h\}} \mathcal{P}^{f^{h'^*}}$ . Since  $\tilde{\mathcal{P}}^h \vee_{h' \in \mathbf{H} \setminus \{h\}} \mathcal{P}^{f^{h'^*}} = \tilde{\mathcal{P}}^h$ , while  $\mathcal{P}^{f^{h^*}} \vee_{h' \in \mathbf{H} \setminus \{h\}} \mathcal{P}^{f^{h'^*}} = \mathcal{P}^*$ , the partition  $\tilde{\mathcal{P}}^h$  is at least as fine as the partition  $\mathcal{P}^*$ .

iv) Since the function  $f^{h^*}$  is measurable with respect to the partition  $\mathcal{P}^*$ , it is measurable and optimal with respect to the partition  $\tilde{\mathcal{P}}^h$ , and the partition  $\tilde{\mathcal{P}}^h$  is at least as fine as the partition  $\mathcal{P}^*$ , the function  $f^{h^*}$  is measurable and optimal with respect to the partition  $\mathcal{P}^*$ .

This completes the argument. ■

**Proposition 2.** A Nash equilibrium where the information of individuals is essentially symmetric with respect to a partition  $\hat{\mathcal{P}}$  is an equilibrium for the game with rational expectations as long as, for any individual, the partition  $\hat{\mathcal{P}}$  is (i) at least as fine as the partition  $\mathcal{P}^h \vee_{h' \in \mathbf{H}} \mathcal{P}^{f^{h'^*}}$  or (ii) at least as coarse as the partition  $\mathcal{P}^h$ .

**Proof:** Let  $f^{\mathbf{H}^*} = (\dots, f^{h^*}, \dots)$  be a Nash equilibrium for the game  $\mathbf{G}_{(\mathbf{S}, \mathcal{P}^{\mathbf{H}})}$  with private information, and suppose there is a partition,  $\hat{\mathcal{P}}$ , such that  $f^{\mathbf{H}^*}$  is a Nash equilibrium for the game  $\mathbf{G}_{(\mathbf{S}, \hat{\mathcal{P}})}$  with symmetric information. We shall show that if, for an individual, the partition  $\hat{\mathcal{P}}$  is at least as fine as the partition  $\mathcal{P}^h \vee_{h' \in \mathbf{H}} \mathcal{P}^{f^{h'^*}}$  or at least as coarse as the partition  $\mathcal{P}^h$ , the act  $f^{h^*}$  is a solution to the optimization problem with respect to the partition  $\mathcal{P}^h \vee_{h' \in \mathbf{H}} \mathcal{P}^{f^{h'^*}}$ , and thus  $f^{\mathbf{H}^*}$  is a Nash equilibrium for the game  $\mathbf{G}_{(\mathbf{S}, \mathcal{P}^{\mathbf{H}})}$  with rational expectations.

For an individual, if  $\hat{\mathcal{P}} \subseteq \mathcal{P}^h \vee_{h' \in \mathbf{H}} \mathcal{P}^{f^{h'^*}}$ , the relevant partitions are ordered as

$$\hat{\mathcal{P}} \subseteq \mathcal{P}^h \vee_{h' \in \mathbf{H}} \mathcal{P}^{f^{h'^*}} \subseteq \mathcal{P}^h \vee_{h' \in \mathbf{H} \setminus \{h\}} \mathcal{P}^{f^{h'^*}} \subseteq \mathcal{P}^h$$

while, if  $\mathcal{P}^h \subseteq \hat{\mathcal{P}}$ , since  $\hat{\mathcal{P}} \subseteq \mathcal{P}^{f^{h'^*}}$ ,

$$\mathcal{P}^h \vee_{h' \in \mathbf{H} \setminus \{h\}} \mathcal{P}^{f^{h'^*}} = \mathcal{P}^h.$$

In either case, since  $f^{h^*}$  is measurable and optimal with respect to the partition  $\mathcal{P}^h$  as well as the partition  $\hat{\mathcal{P}}$ , it is measurable and optimal with respect to the partition  $\mathcal{P}^h \vee_{h' \in \mathbf{H} \setminus \{h\}} \mathcal{P}^{f^{h'^*}}$ .

This completes the argument. ■

A Nash equilibrium where the information of individuals is symmetric, but with respect to a partition  $\hat{\mathcal{P}}$  which, for some individuals, fails to be either as fine as the partition  $\mathcal{P}^h \vee_{h' \in \mathbf{H}} \mathcal{P}^{f^{h'}}$  or as coarse as the partition  $\mathcal{P}^h$ , need not be an equilibrium with rational expectations.

**Example 1.**

Consider a game with  $\mathbf{H} = \{1, 2\}$ ,  $\mathbf{A}^1 = \{T, B\}$ ,  $\mathbf{A}^2 = \{L, R\}$ ,  $\mathbf{S} = \{1, 2, 3\}$ ,  $\mathcal{P}^1 = \{\{1\}, \{2, 3\}\}$ ,  $\mathcal{P}^2 = \{\{1\}, \{2\}, \{3\}\}$ ,  $\pi = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ , and payoffs  $u^h(f^1, f^2) = \sum_{s \in \mathbf{S}} \pi(s) u_s^h(f^1(s), f^2(s))$  described by the matrices

	$L$	$R$
$T$	5, 1	5, 2
$B$	1, 1	1, 2
	$s = 1$	

	$L$	$R$
$T$	2, 1	2, 2
$B$	3, 1	3, 2
	$s = 2$	

	$L$	$R$
$T$	3, 2	3, 1
$B$	1, 2	1, 1
	$s = 3$	

At a Nash equilibrium, player 1 plays  $T$  in every state, while player 2 plays  $R$  in states 1 and 2, and  $L$  in state 3. Information is essentially symmetric with respect to the partition  $\hat{\mathcal{P}} = \{\{1, 2\}, \{3\}\}$ . This is not a rational expectations equilibrium; player 1, with information  $\mathcal{P}^1 \vee \mathcal{P}^{f^2} = \{\{1\}, \{2\}, \{3\}\}$ , would play  $T$  in states 1 and 3, and  $B$  in state 2. ■

**3. Common knowledge and rational expectations**

Let the information of an individual  $h \in \mathbf{H}$  be described by the partition  $\mathcal{Q}^h$  of the set,  $\mathbf{S}$ , of states of the world.

At a state of the world,  $\bar{s}$ , an individual knows an event  $\mathbf{E} \subseteq \mathbf{S}$  if and only if  $\mathcal{Q}^h(\bar{s}) \subseteq \mathbf{E}$ . The states of the world at which the individual knows  $\mathbf{E}$  is

$$K^h(\mathbf{E}) = \{s \in \mathbf{S}, \text{ such that } \mathcal{Q}^h(s) \subseteq \mathbf{E}\}.$$

Equivalently,  $\bar{s} \notin K^h(\mathbf{E})$ , the individual does not know  $\mathbf{E}$  at  $\bar{s}$ , if and only if  $s' \notin \mathbf{E}$  for some  $s' \in \mathcal{Q}^h(\bar{s})$ .

Suppose there exist finite sequences of states of the world,  $s_n, s_{n-1}, \dots, s_1 = \bar{s}$  and individuals,  $h_n, h_{n-1}, \dots, h_1$ , not necessarily distinct, such that  $s_n \in \mathbf{Q}^{h_{n-1}}(s_{n-1})$ ,  $\dots, s_2 \in \mathbf{Q}^{h_1}(\bar{s})$  and also  $s' \in \mathbf{Q}^{h_n}(s_n) \setminus \mathbf{E}$ . Since  $s' \in \mathbf{Q}^{h_n}(s_n) \setminus \mathbf{E}$ ,  $s_n \notin K^{h_n}(\mathbf{E})$ . But then, since  $s_n \in \mathbf{Q}^{h_{n-1}}(s_{n-1}) \setminus K^{h_n}(\mathbf{E})$ ,  $s_{n-1} \notin K^{h_{n-1}}(K^{h_n}(\mathbf{E}))$ . Continuing in this manner, we obtain that  $\bar{s} \notin K^{h_1}(K^{h_2}(\dots, K^{h_n}(\mathbf{E})))$ : individual  $h_1$  does not know that individual  $h_2$  knows that ... individual  $h_{n-1}$  knows that individual  $h_n$  knows  $\mathbf{E}$ . Thus, for any finite sequence of individuals  $h_n, h_{n-1}, \dots, h_1$ , individual  $h_1$  knows that  $h_2$  knows that ...  $h_{n-1}$  knows that  $h_n$  knows  $\mathbf{E}$  at  $\bar{s}$  if and only if for any sequence of states of the world  $s_n, s_{n-1}, \dots, s_2, s_1 = \bar{s}$  such that  $s_n \in \mathbf{Q}^{h_{n-1}}(s_{n-1}), \dots, s_2 \in \mathbf{Q}^{h_1}(\bar{s}), \mathbf{Q}^{h_n}(s_n) \subseteq \mathbf{E}$ ; equivalently, the event  $\mathbf{E}$  contains  $\underline{\mathbf{Q}}(\bar{s})$ , the element of the meet or finest common coarsening of the individual partitions.

At a state of the world  $\bar{s} \in \mathbf{S}$ , an event  $\mathbf{E} \subseteq \mathbf{S}$  is common knowledge if and only if

$$\underline{\mathbf{Q}}(\bar{s}) \subseteq \mathbf{E}, \text{ where } \underline{\mathbf{Q}} = \bigwedge_{h \in \mathbf{H}} \mathbf{Q}^h.$$

A function,  $f$ , with domain the set of states of the world,  $\mathbf{S}$ , is common knowledge at  $\bar{s}$  if and only if the event  $f^{-1}(f(\bar{s})) = \{s \in \mathbf{S}, \text{ such that } f(s) = f(\bar{s})\}$  is common knowledge at  $\bar{s}$ . A function,  $f$ , is common knowledge if and only if it is common knowledge at all  $\bar{s} \in \mathbf{S}$ .

**Corollary 1.**(i) At a Nash equilibrium for the game with rational expectations, the acts of all individuals are common knowledge.(ii) A Nash equilibrium where the acts of all players are common knowledge is a Nash equilibrium for the game with rational expectations.

**Proof:** We argue in steps

(i) Let  $f^{\mathbf{H}^*} = (\dots, f^{h^*}, \dots)$  be Nash equilibrium profile of acts for the game with rational expectations. The information partition of an individual at equilibrium is  $\tilde{\mathcal{P}}^h = \mathcal{P}^h \vee_{h' \in \mathbf{H}/\{h\}} \mathcal{P}^{f^{h'^*}}$ , which is at least as fine as the partition  $\vee_{h' \in \mathbf{H}/\{h\}} \mathcal{P}^{f^{h'^*}}$ . It follows that the meet of the individuals partitions at the equilibrium is at least as fine as the partition  $\mathcal{P}^* = \vee_{h \in \mathbf{H}} \mathcal{P}^{f^{h^*}}$ . Since the act  $f^{h^*}$  is measurable with respect to the partition  $\mathcal{P}^*$ , the result follows.

(ii) Let  $f^{\mathbf{H}^*} = (\dots, f^{h^*}, \dots)$  be a Nash equilibrium profile of acts, where the acts of

all players are common knowledge. It follows that  $f^{h^*}$  is measurable with respect to the meet of private individual partitions,  $\underline{\mathcal{P}} = \bigwedge_{h \in \mathbf{H}} \mathcal{P}^h$ . Since  $f^{h^*}$  is optimal with respect to the partition  $\mathcal{P}^h \subseteq \underline{\mathcal{P}}$ , by proposition 2 the result follows. ■

Corollary 1(ii) and Proposition 1 immediately imply that at a Nash equilibrium in which acts happen to be common knowledge information is essentially symmetric. This is an instance of the theorem that “common knowledge of actions negates asymmetric information about events” in Geanakoplos (1989).

Proposition 1 plays the same role with respect to this theorem that the result in Geanakoplos and Polemarchakis (1982) plays with respect to the theorem in Aumann (1976): even if, “to begin with” information is not symmetric and acts are not common knowledge, “eventually” acts are common knowledge and information is symmetric. The process of communication is not explicit. Rather, it is embedded in the definition of a game with rational expectations.

**Example 2.** (Geanakoplos and Polemarchakis (1982))

The opinion game is a game with uncertainty and private information,  $\mathbf{O}_{(\mathbf{S}, \mathcal{P}^{\mathbf{H}})} = \{\mathbf{H}, \mathbf{S}, (\mathbf{A}^h, u^h, \mathcal{P}^h) : h \in \mathbf{H}\}$ , where  $\mathbf{A}^h = \mathbf{R}$ , and  $u^h(f^{\mathbf{H}}) = -\sum_{s \in \mathbf{S}} \pi(s)(f^h(s) - x(s))^2 = -E_{\pi}(f^h - x)^2$ , for a common prior probability measure,  $\pi$ , on the set of states of the world,  $\mathbf{S}$ . Individuals guess the value of the random variable  $x : \mathbf{S} \rightarrow \mathbf{R}$ .

At a Nash equilibrium,  $f^{\mathbf{H}^*}$ , of this game, each player chooses the conditional expectation of  $x$ , given his private information at each state of the world:  $f^{h^*}(s) = E_{\pi}(x | \mathbf{P}^h(s))$ , for  $s \in \mathbf{S}$  and  $h \in \mathbf{H}$ .

Individuals may disagree, due to differences in information. At a Nash equilibrium with rational expectations individuals choose a profile  $f^{\mathbf{H}^*}$  which, by Proposition 1, is also a Nash equilibrium for the opinion game with symmetric information  $\mathbf{O}_{(\mathbf{S}, \mathcal{P}^*)} = (\mathbf{H}, \mathbf{S}, (\mathbf{A}^h, u^h, \mathcal{P}^*, \pi) : h \in \mathbf{H})$ , where  $\mathcal{P}^* = \bigvee_{h \in \mathbf{H}} \mathcal{P}^h$ . That is, at a Nash equilibrium with rational expectations,  $f^{h^*}(s) = E_{\pi}(x | \mathbf{P}^*(s))$ , for  $s \in \mathbf{S}$  and  $h \in \mathbf{H}$ .

Individuals “agree” in the opinion game with rational expectations

Consider an opinion game with  $\mathbf{S} = \{1, 2, \dots, 9\}$ ,  $\mathbf{H} = \{1, 2\}$ ,  $\mathcal{P}^1 = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}$ ,  $\mathcal{P}^2 = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9\}\}$ ,  $\pi = (\frac{1}{9}, \dots, \frac{1}{9})$ , and

$$x(s) = \begin{cases} 17, & \text{if } s = 1, 5, 9, \\ -7, & \text{if } s = 2, 3, 4, 6, 7, 8. \end{cases}$$

The unique Nash equilibrium profile of acts is

$$f^{1*}(s) = 1, \\ f^{2*}(s) = \begin{cases} 17, & \text{if } s = 9, \\ -1, & \text{if } s \neq 9. \end{cases}$$

In the game with rational expectations this would not be a Nash equilibrium: player 1 could use the information revealed by  $f^{2*}$  to separate states (7, 8) from (9), which would render  $f^{1*}$  suboptimal.

Alternatively, the profile of acts

$$f^{1**}(s) = f^{2**}(s) = \begin{cases} 1, & \text{if } s = 1, 2, 3, \\ -7, & \text{if } s = 4, 7, 8, \\ 5, & \text{if } s = 5, 6, \\ 17, & \text{if } s = 9, \end{cases}$$

is a Nash equilibrium for the game with rational expectations. At this equilibrium information is essentially symmetric.

That at the Nash equilibrium for the game with rational expectations the equilibrium acts of individuals coincide, the individuals “agree,” follows simply from the symmetry of the game and the symmetry of information at equilibrium. ■

#### 4. Speculation

In a game,  $\mathbf{G} = (\mathbf{H}, (\mathbf{F}^h, u^h) : h \in H)$ , a subset of individuals  $\hat{\mathbf{H}} \subseteq \mathbf{H}$  is a veto coalition if and only if each individual in the coalition has a veto act  $z^h \in \mathbf{F}^h$ , which guarantees to him a given level of utility  $\bar{u}^h$  no matter what the other individuals do:

$$u^h(z^h, f^{H \setminus \{h\}}) \geq \bar{u}^h, \quad f^{\mathbf{H} \setminus \{h\}} \in F^{\mathbf{H} \setminus \{h\}}, \quad h \in \hat{\mathbf{H}},$$

and such that any feasible profile  $f^{\mathbf{H}}$  which gives to each individual in the coalition at least  $\bar{u}^h$ , gives exactly  $\bar{u}^h$ :

$$u^h(f^{\mathbf{H}}) \geq \bar{u}^h \Rightarrow u^h(f^{\mathbf{H}}) = \bar{u}^h, \quad h \in \hat{\mathbf{H}}.$$

At a Nash equilibrium of a game with a veto coalition, all individuals in the coalition attain the level of utility associated with their veto acts.

With uncertainty,  $\mathbf{F}^h = \{f^h : \mathbf{S} \rightarrow \mathbf{A}^h\}$ .

With private information there is speculation at a profile  $f^{\mathbf{H}}$  if the event  $\mathbf{E} = \{s \in \mathbf{S} : \sum_{s' \in \mathbf{P}^h(s)} u_{s'}^h(f^{\mathbf{H}}(s')) \geq \sum_{s' \in \mathbf{P}^h(s)} u_{s'}^h(z^{\hat{\mathbf{H}}}(s'), f^{\mathbf{H} \setminus \hat{\mathbf{H}}}(s')), h \in \hat{\mathbf{H}}, \text{ with some strict inequality } \}$  realizes.

Speculation can occur at a Nash equilibrium of the game with rational expectations.

**Example 3.**

Consider a game with  $\mathbf{H} = \{1, 2\}$ ,  $\mathbf{A}^1 = \{T, B\}$ ,  $\mathbf{A}^2 = \{L, R\}$ ,  $\mathbf{S} = \{1, 2\}$ ,  $\mathcal{P}^1 = \{\{1\}, \{2\}\}$ ,  $\mathcal{P}^2 = \{\{1, 2\}\}$ ,  $\pi = (\frac{1}{2}, \frac{1}{2})$ , and payoffs  $u^h(f^1, f^2) = \sum_{s \in \mathbf{S}} \pi(s) u_s^h(f^1(s), f^2(s))$  described by the matrices

	<i>L</i>	<i>R</i>
<i>T</i>	1, -1	1, 0
<i>B</i>	1, 0	0, 0
	<i>s = 1</i>	

	<i>L</i>	<i>R</i>
<i>T</i>	-1, 1	-1, 0
<i>B</i>	-1, 0	0, 0
	<i>s = 2</i>	

Here the veto coalition contains both individuals. Choosing *B* in both states, player 1 can guarantee an expected utility of 0, and player 2 can do the same if he plays *R*. On the other hand, player 1 playing *T* in both states and player 2 playing *L* in both states is a Nash equilibrium, and also a Nash equilibrium of the game with rational expectations. At state 1 there is speculation: player 1 strictly prefers the equilibrium profile to the veto profile, and player 2 is indifferent, given his information. The equilibrium acts are common knowledge, but there is no common knowledge of speculation: in state 2 player 1 prefers the veto profile. ■

For a profile of acts,  $f^{\mathbf{H}} = (\dots, f^h, \dots)$ , not necessarily a Nash equilibrium, and for a state of the world,  $\bar{s}$ , the event  $\mathbf{E}$  cannot be common knowledge. Indeed, if the meet of the individuals' partitions is denoted by  $\mathcal{P}$ , common knowledge of speculation

at  $\bar{s}$  means that  $\sum_{s' \in \mathbf{P}(\bar{s})} u_{s'}^h(f^{\mathbf{H}}(s')) \geq \sum_{s' \in \mathbf{P}(\bar{s})} u_{s'}^h(z^{\hat{\mathbf{H}}}(s'), f^{\mathbf{H} \setminus \hat{\mathbf{H}}}(s')), h \in \hat{\mathbf{H}}$ , with some strict inequality. But then, by choosing acts which coincide with  $f^h$  on  $\mathbf{P}(\bar{s})$  and with  $z^h$  on  $\mathbf{S} \setminus \mathbf{P}(\bar{s})$ , all individuals in  $\hat{\mathbf{H}}$  would be at least as well off as when they choose their veto act, and some strictly better off, which is a contradiction. This is the argument of Milgrom and Stokey (1982) that speculation cannot be common knowledge.

To obtain no speculation as a consequence of Nash equilibrium we need to consider a more restricted class of games.

A subset of individuals  $\hat{\mathbf{H}} \subseteq \mathbf{H}$  is a strong veto coalition if and only if each individual in the coalition has a veto act  $z^h \in \mathbf{F}^h$ , such that, in any feasible profile  $f^{\mathbf{H}}$  which gives to each individual in the coalition at least  $\bar{u}^h$ , each plays exactly  $z^h$ :

$$u^h(f^{\mathbf{H}}) \geq \bar{u}^h \Rightarrow f^h = z^h, h \in \hat{\mathbf{H}}.$$

At a Nash equilibrium of a game with a strong veto coalition, all individuals in the coalition play their veto acts.

The argument extends immediately to games with uncertainty as well as to games with uncertainty and private information.

In particular, speculation never realizes if  $f^{\mathbf{H}}$  is a Nash equilibrium profile. This is the argument of Geanakoplos (1989) that Nash equilibrium suffices to prevent betting or speculation in games with a strong veto coalition.

The notion of veto game is very demanding; the veto act  $z^{\mathbf{H}}$  is required to dominate all acts. With a weaker notion, for example with  $\mathbf{F}^h = \{f^h : \mathbf{S} \rightarrow \mathbf{A}^h, \text{ such that } f^h \text{ is measurable with respect to } \mathcal{P}^h\}$ , speculation may well happen at equilibrium, and be common knowledge with respect the equilibrium information  $\mathcal{P}^*$ . To see this, let  $z^1 = (T, T, T)$  and  $z^2 = (R, R, L)$  in example 1.

If every individual has a veto strategy,  $\hat{\mathbf{H}} = \mathbf{H}$ , and if the veto acts of all individuals are measurable with respect to some partition,  $\mathcal{P}^*$ , at least as coarse as their private information,  $\mathcal{P}^h \subseteq \mathcal{P}^*$ , the game has a Nash equilibrium,  $z^{\mathbf{H}}$ , at which information is essentially symmetric and which, as a consequence is a Nash equilibrium for the game with rational expectations.

**Example 4.**

Consider two individuals betting on the value of a real random variable  $x : \mathbf{S} \rightarrow \mathcal{R}$ . At each state of the world, an individual declares if he accepts the bet,  $a^h(s) = 1$ , or not,  $a^h(s) = 0$ . If both accept, individual  $h = 2$  transfers  $x(s)$  dollars to individual  $h = 1$ . If one of the two individuals does not accept it, the bet doesn't take place. The payoff matrix at a state  $s$  is thus

		$a^2$			
		1		0	
	$a^1$	1	$x(s), -x(s)$	0	0, 0
		0	0, 0	0	0, 0

$, s \in \mathbf{S}.$

Each individual chooses a strategy  $f^h \in \mathbf{F}^h = \{f : \mathbf{S} \rightarrow \{0, 1\}\}$ , such that  $f$  is measurable with respect to  $\mathcal{P}^h$  to maximize his expected payoff.

The betting game is thus  $\mathbf{B}_{(\mathbf{S}, \mathcal{P}^H)} = \{\mathbf{H}, \mathbf{S}, (\mathbf{A}^h, u^h, \mathcal{P}^h, \pi)\}$ , where  $\mathbf{H} = \{1, 2\}$ ,  $\mathbf{A}^h = \{0, 1\}$ , and  $u^h(f^H) = (-1)^{h+1} \sum_{s \in \mathbf{S}} f^1(s) f^2(s) x(s) \pi(s)$ .

In particular, consider a betting game with  $\mathbf{S} = \{1, 2, \dots, 9\}$ ,  $\mathcal{P}^1 = \{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}\}$ ,  $\mathcal{P}^2 = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9\}\}$ ,  $\pi = (\frac{1}{9}, \dots, \frac{1}{9})$ , and

$$x(s) = \begin{cases} 17 & s = 1, 5, 9 \\ -7 & s = 2, 3, 4, 6, 7, 8. \end{cases}$$

The information structure is exactly the same as in the example of guessing game discussed above, but the two individuals are betting on the value of  $x$  instead of simply giving their opinions about it.

It is easy to see that always refusing the bet is an equilibrium strategy for both players. In fact, when  $s = 9$ , individual 2 knows the true state and faces the matrix

		$a^2$		
		1	0	
	$a^1$	17, -17	0, 0	, $s = 9$ .
	0	0, 0	0, 0	

At a Nash equilibrium, individual 2 chooses not to bet at  $s = 9$ . It follows that individual 1 chooses not to bet at  $s = 7, 8, 9$ , individual 2 chooses not to bet at  $s = 5, 6, 7, 8$  and so on. Individual 1 is indifferent between betting or not at  $s = 1, 2, 3$ ; in both cases the two individuals have the same payoff,  $u^1 = u^2 = 0$ .

The information of the individuals at the Nash equilibrium  $f^{1*}(s) = f^{2*}(s) = 0, s \in S$ , is essentially symmetric: constant acts are measurable with respect to the trivial partition  $\mathcal{P}^* = \{S\}$ . This is also a Nash equilibrium for the game with rational expectations, and acts are common knowledge among individuals at equilibrium. ■

It remains an open question to characterize the class games with uncertainty and private information such that at a Nash equilibrium information is essentially symmetric. For such games, the distinction between a Nash equilibrium and a Nash equilibrium for the game with rational expectations vanishes.

### 5. Speculation at a competitive equilibrium

A competitive economy with uncertainty and private information is

$$E_{(S, \mathcal{P}^H)} = \{H, S, (Z^h, \mathcal{P}^h, u^h) : h \in H\}.$$

The private information of an individual is  $\mathcal{P}^h$ , and  $\mathcal{P}^H$  is the profile of private information, across individuals.

An elementary net trade or an elementary act for an individual is  $z^h \in Z^h$ , and a net trade, across states of the world, is  $f^h \in F^h = \{f^h : S \rightarrow Z^h\}$ . The utility

function of the individual over net trades is  $u^h$ .

An allocation or a profile of acts is a collection,  $f^{\mathbf{H}} = \{f^h : h \in \mathbf{H}, \text{ such that } f^h : \mathbf{S} \rightarrow \mathbf{Z}^h\}$ , of net trades. It is feasible if and only if  $\sum_{h \in \mathbf{H}} f^h(s) = 0$ , for  $s \in \mathbf{S}$ . It is pareto optimal if and only if there does not exist another allocation,  $f^{\mathbf{H}'}$ , which is feasible and such that  $u^h(f^{h'}) \geq u^h(f^h)$ , for  $h \in \mathbf{H}$ , with some strict inequality.

We say that there is speculation at an allocation of net trades,  $f^{\mathbf{H}}$ , if and only if the no-trade allocation,  $\bar{f}^{\mathbf{H}} = 0$ , is pareto optimal, and the event  $\mathbf{E} = \{s \in \mathbf{S}, \text{ such that } \sum_{s' \in \mathcal{P}^h(s)} u_{s'}^h(f^h(s)) \geq \sum_{s' \in \mathcal{P}^h(s)} u_{s'}^h(0), h \in \mathbf{H}, \text{ with some strict inequality}\}$  realizes.

Elementary commodity prices are  $\pi \in \Delta$ , the unit simplex, and commodity prices are  $p : \mathbf{S} \rightarrow \Delta$ .

In an economy with rational expectations, individuals refine their information with the information revealed by prices.

The information revealed by prices  $p$  is  $\mathcal{P}^p$ .

A competitive equilibrium for an economy with rational expectations, Radner (1979), is a pair,  $(p^*, f^{\mathbf{H}^*})$ , of prices and a feasible allocation, such that, for an individual,  $f^{h^*}$  is a solution to the maximization of utility over the set  $\mathbf{B}_{(\mathbf{S}, \mathcal{P}^h \vee \mathcal{P}^{p^*})}^h(p^*) = \{f : \mathbf{S} \rightarrow \mathbf{Z}^h, \text{ such that } f \text{ is measurable with respect to } \mathcal{P}^h \vee \mathcal{P}^{p^*}, \text{ and } p^*(s)f(s) \leq 0, s \in \mathbf{S}\}$ .

Associated with an economy, Debreu (1952), there is a generalized game with uncertainty and private information

$$\Gamma(\mathbf{E}_{(\mathbf{S}, \mathcal{P}^{\mathbf{H}})}) = \{\mathbf{H}^0, \mathbf{S}, (\mathbf{A}^h, u^h, \mathcal{P}^h, \psi^h) : h \in \mathbf{H}^0\}.$$

Individuals are  $h \in \mathbf{H}^0 = \{0\} \cup \mathbf{H}$ , and individual  $h = 0$  is the ‘‘auctioneer.’’ The set of elementary acts for the auctioneer is  $\mathbf{A}^0 = \Delta$ , the domain of elementary commodity prices, his utility function is  $u^0(f^{\mathbf{H}^0}) = \sum_{h \in \mathbf{H}} \sum_{s \in \mathbf{S}} p(s)f^h(s)$ , his information is complete:  $\mathcal{P}^0 = \{\{s\} : s \in \mathbf{S}\}$ , and his choices are not constrained by the acts of other players:  $\psi^0(f^{\mathbf{H}^0 \setminus \{0\}}) = \mathbf{F}^0 = \{p : \mathbf{S} \rightarrow \Delta\}$ . For an individual, the set of elementary acts is the set of elementary net trades,  $\mathbf{A}^h = \mathbf{Z}^h$ .

To prove that speculation cannot occur at a competitive equilibrium of the econ-

omy with rational expectations, let the utility functions of individuals be separable across states of the world,  $u^h(f^h) = \sum_{s \in \mathbf{S}} u_s^h(f^h(s))$ , and strictly quasi concave, so that every veto coalition is also a strong veto coalition.

The optimization problem of an individual at given prices  $p$  and at a state of the world  $s$  is

$$\max_{f^h(s) \in \mathbf{B}_s^h(p)} \sum_{s' \in \mathbf{Q}^h(s)} u_{s'}^h(f^h(s))$$

with

$$\mathbf{B}_s^h(p) = \{z^h \in \mathbf{Z}^h \text{ such that } p(s)z^h \leq 0\}$$

and  $\mathbf{Q}^h = \mathcal{P}^h$  or, in the economy with rational expectations,  $\mathbf{Q}^h = \mathcal{P}^h \vee \mathcal{P}^p$ .

For given prices  $p$ ,  $\mathbf{B}^h(s) = \mathbf{B}_s^h(p)$  defines a correspondence from states of the world to elementary acts. If this correspondence is measurable with respect to the information partition of the individual, the solutions to the individual optimization problems coincide for  $s$  and  $s'$  with  $\mathbf{Q}^h(s) = \mathbf{Q}^h(s')$ , and yield unambiguously a solution to the ex-ante optimization problem. This is the case in the economy with rational expectations. In the economy with rational expectations, solutions of the ex-post and ex-ante problems coincide, and are Nash equilibria of a well defined generalized game.

At a competitive equilibrium allocation in the economy with rational expectations, speculation is thus precluded. It suffices to observe that a competitive equilibrium allocation is a Nash equilibrium allocation, while pareto optimality of  $\bar{f}^{\mathbf{H}}$  guarantees that not to trade is a veto profile of acts for individuals other than the auctioneer. This is the argument of Kreps (1977) and Tirole (1982).

In the economy without rational expectations, the budget correspondence need not be measurable with respect to the individual information partition; solutions need not coincide if  $\mathbf{Q}^h(s) = \mathbf{Q}^h(s')$ , and need not yield a solution to the ex-ante optimization problem. A competitive equilibrium need not be a Nash equilibrium of any well defined generalized game and speculation might occur.

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