THE EX-ANTE NON-OPTIMALITY OF THE DEMPSTER-SCHAFER UPDATING RULE FOR AMBIGUOUS BELIEFS

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ABSTRACT

The most widely used updating rule for non-additive probabilities is the Dempster-Schafer rule. Schmeidler and Gilboa have developed a model of decision making under uncertainty based on non-additive probabilities, and in their paper "Updating Ambiguous Beliefs" they justify the Dempster-Schafer rule based on a maximum likelihood procedure. This note shows in the context of Schmeidler-Gilboa preferences under uncertainty, that the Dempster-Schafer rule is in general not ex-ante optimal. This contrasts with Brown's result that Bayes' rule is ex-ante optimal for standard Savage preferences with additive probabilities.

1 INTRODUCTION

The standard representation in Economics of rational behavior under uncertainty is Savage's model. Savage's axiomatic derivation does not address the problem of how to update beliefs; nevertheless many considerations point to using Bayes' rule in his model. Brown(1976) showed that Bayes' rule is optimal in the sense that it maximizes ex-ante expected utility.

Schmeidler and Gilboa in a series of papers (Schmeidler(1982,1989), Gilboa(1987), Gilboa and Schmeidler(1989)) present a theory which is an alternative to Savage's theory of choice under uncertainty. In their theory of choice the individuals maximize the expected utility with respect to a non-additive probability, in the spirit of the belief function literature (Schafer(1976)). An application of this theory to portfolio selection may be found in Dow and Werlang(1992).

Again, the original axiomatic development of the model did not address the question of updating. The Dempster-Schafer rule (DS for short) seems the natural candidate for the updating rule, although the situation is not as clear cut as in the case of Savage's model (see Wasserman and Kadane(1990)). Gilboa and Schmeidler(1992) give a justification of DS based on a maximum likelihood procedure. In this context it makes sense to ask whether DS is ex-ante optimal. In this note we give an example which shows that this is not the case.

2 EX-ANTE OPTIMALITY OF BAYES' RULE

Brown's result that Bayes' rule is ex-ante optimal may be expressed in formal terms as follows. Let Ω be the set of possible states of the world, assumed finite for simplicity, and let P be the subjective probability distribution defined on Ω. Assume $\Pi = \{A_1, A_2, ..., A_n\}$ is a partition of Ω, with $P(A_i) > 0$ for all i=1, ...,n. An updating rule is a function $Q: 2^\Omega \times \Pi \rightarrow [0,1]$, with the property that given $A \in \Pi$, $Q(\cdot, A)$ is a probability distribution with $Q(A, A) = 1$. We write $Q(\omega, A)$ for $Q(\{\omega\}, A)$ if $\omega \in \Omega$. The interpretatation of $Q(E, A_i)$ is the probability of occurrence of the event E, given that the cell A_i of the partition Π has occurred. It should be clear that, in the way we defined Q, it also depends on the partition. Bayes' rule of updating is $Q(E, A) = \mathbf{B}(E, A) = P(E \cap A)/P(A)$ for any event A with $P(A) > 0$, which is independent of the partition. An act is a function from the set of states Ω into the set of outcomes O. Let $u: O \rightarrow \Re$ be the utility function. The ex-ante utility of an act α is $\Sigma_{\omega \in \Omega} P(\omega)u(\alpha(\omega))$. Given that the set A_i in the partition occurred, and the updating rule is Q, the ex-post utility is $\Sigma_{\omega \in A_i} Q(\omega, A_i)u(\alpha(\omega))$. Observe that it is irrelevant whether or not the sum is over A_i or over Ω, because, by definition, $Q(\omega, A_i) = 0$ if ω does not belong to A_i. Hence it makes sense to define the ex-ante utility of an act α given the updating rule is Q: $U(\alpha, Q) = \Sigma_{i=1,...,n} P(A_i)[\Sigma_{\omega \in A_i} Q(\omega, A_i)u(\alpha(\omega))]$. Assume that information is revealed prior to the choice of the act. Given that the set $A_i \in \Pi$ occurred, and that the updating rule Q is used, the individual chooses an act $\alpha_i(Q)$ which maximizes the ex-post expected utility i.e. $\Sigma_{\omega \in A_i} Q(\omega, A_i)u(\alpha_i(Q)(\omega)) \geq \Sigma_{\omega \in A_i} Q(\omega, A_i)u(\alpha(\omega))$, for all acts α (since it makes no difference whether or not acts are defined only on A_i, we assume they are defined only on A_i for ease of notation). The following result is easily proved (Brown(1976)):

Theorem Let $\alpha(Q) = (\alpha_1(Q), \alpha_2(Q), ..., \alpha_n(Q))$ be the act defined as the combination of all ex-post optimal acts, given the updating rule Q. Then for any partition Π, it follows that Bayes' rule (defined by **B** above) is the updating rule which yields the maximum ex-ante expected utility, that is to say, $U(\alpha(\mathbf{B}),\mathbf{B}) \geq U(\alpha(Q),Q)$ for any updating rule Q.

3 AN EXAMPLE

The following example shows that this property does not hold for the Dempster-Schafer rule applied to the Schmeidler-Gilboa model.

Example Let $\Omega = \{1,2,3\}$, and $\Pi = \{\{1,2\},\{3\}\}$. Let also the set of outcomes be $\mathfrak{R}$, and $u(x) = x$ for all $x \in \mathfrak{R}$. Assume the non-additive probability distribution P (which is also a belief function) is given by $P_1 = 0.1$, $P_2 = 0$, $P_3 = 0.05$, $P_{13} = 0.3$, $P_{23} = 0.65$, $P_{12} = 0.3$ ($P_{ij} = P(\{i,j\})$) . Consider the act α given by $\alpha(1) = -2$, $\alpha(2) = 3$, $\alpha(3) = 1$ and the act β given by $\beta(1) = \beta(2) = 0$, $\beta(3) = 1$. By making use of the DS updating rule, we have $E_P[\alpha \mid \{1,2\}] = 22/19$ and $E_P[\alpha \mid \{3\}] = 1$. Clearly, $E_P[\beta \mid \{1,2\}] = 0$ and $E_P[\beta \mid \{3\}] = 1$. Thus, given the occurrence of any cell of the partition and the use of DS rule, act α is always weakly preferred to act β. In fact, in the set $\{1,2\}$ it is strictly preferred. Therefore, if we assume that the set of available acts is $\{\alpha,\beta\}$, it turns out that α is the combination of optimal ex-post choices given DS updating rule. However, ex-ante we have that $E[\alpha] = -0.05 < 0.05 = E[\beta]$, which means that act β is strictly preferred ex-ante to act α. Observe that the conditioning rule Q uniquely defined by the conditions: $Q(1,\{1,2\}) = Q(2,\{1,2\}) = 1/2$ and $Q(3,\{3\}) = 1$, gives act β as the combination of ex-post optimal acts (in the set $\{\alpha,\beta\}$). Hence, the updating rule Q yields higher ex-ante payoffs than DS rule. This shows that DS updating rule is not ex-ante optimal in general. The phenomenon of preference reversal with the partial revelation of information was noticed before in Dow, Madrigal and Werlang(1989), in an example similar to the one presented above.

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