INTENTIONAL PARADOXES AND AN INDUCTIVE THEORY OF PROPOSITIONAL QUANTIFICATION

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ABSTRACT

Quantification over propositions is a necessary component of any theory of attitudes capable of providing a semantics of attitude ascriptions and a sophisticated system of reasoning about attitudes. There appear to be two general approaches to propositional quantification. One is developed within a first order quantificational language, the other in the language of higher order logic. The first order theory is described in Asher & Kamp (1986), Asher (1988), Asher and Kamp (1989). This paper concentrates on propositional quantification in a higher order framework, the simple theory of types. I propose a method of resolving difficulties noticed by Prior and Thomason with propositional quantification. The method borrows from Kripke's (1975) defintition of truth and results in a partial logic, which I call the *simple theory of partial types* (SPT). SPT offers a tractable, complete logic (with respect to general models) that includes propositional quantification, accomodates a semantics of the attitudes that avoids logical omniscience, and allows for some self-reference.

1.Introduction

Consider the following examples in which there is apparent quantification over propositions.[1]

(1.a) Everything Mary believes is true.

(1.b) Every fact you discover may be relevant.

(1.c) Nothing you have said convinces me.

The question I would like to pose here is a familiar one from analytic philosophy since the turn of the century:[2] what is the logical form underlying this apparent reference and quantification over abstract

[1]I would like to thank Rich Thomason for comments on an earlier version of this paper read at the Logic and Linguistics Meetings in Tuscon AZ 1989.

[2]See for instance Russell's (1903) arguments in *The Principles of Mathematics*. The concern with abstract entities and their logic remained a concern throughout Russell's life.

entities? Two general theories emerge, one a first order theory, the other a higher order theory of quantification.[1] The difficult task for such theories is to develop a coherent theory of quantification over abstract objects that are suitably descriminated to be objects of attitudes. The task is difficult because many attempts to do so have led to paradoxes concerning abstract entities. These paradoxes have bedeviled philosophers and logicians since ancient times.

There are two generally recognized families of paradoxes. One contains paradoxes having to do with sentences and direct quotation contexts like the Liar. Then there are paradoxes of application like the property version of Russell's paradox and the family of associated set theoretic paradoxes (Burali-Forti, Russell, etc). Arthur Prior (1961) and more recently Rich Thomason (1982) have argued that there is a third family of paradoxes, the so called "paradoxes of indirect discourse," which have to do with the nature of propositions or other abstract entities. The category of paradoxes of indirect discourse is potentially very varied. The defining characteristic of a paradox of indirect discourse is that it does not directly involve a quotational context.[2] Here is an example of such a paradox originally due to Jean Buridan, embellished by Prior and Thomason: Suppose Prior is thinking to himself,

(2) Either everything that I am thinking at the present moment or everything that Tarski will think in the next instant, but not both, is false.

Suppose that at the present moment Prior thinks nothing else and at the next moment Tarski thinks that snow is white and nothing else. I'll call this the *Prior situation.* By reasoning that is valid in the simple theory of types, we conclude that Tarski was not able to think that snow is white, a bizarre and unwanted consequence of a logic for belief.

The two theories of quantification dictate two approaches to such intentional paradoxes. Beginning with a representationalist's view of attitudes and abstract entities, one can arrive at a natural formulation in a first order language of what Prior is thinking to himself. This is a congenial perspective to someone committed to a representational theory of attitudes. By exploiting the inductive or semi-inductive techniques used by Kripke (1975), Herzberger (1982) and Gupta (1982) to define truth, one can build a families of models and develop a variety of logics for knowledge and belief.[3] Such a framework assimilates a treatment

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[1]Many people have been suggesting a first order theory of abstract entities in the past few years-- for instance Bealer (1982), Turner (1987), (1989), Aczel (1989). Higher order theories have found advocates like Russell (1901) (1911), Ramsey (1926), Prior (1960) e.g., and others like Fine, Cocchiarella, and Thomason (1980.b), and Menzel. I will use Turner and Thomason as my main sources here, but that is not because I have made a detailed survey of all the proposals.

[2] Other ways of constructing paradoxes of indirect discourse do not depend on direct discourse at all. There are paradoxes of intention (similar to Newcomb's Problem and explored recently by Gaifman) that resemble at least semantic paradoxes. Gaifman's puzzle gives a *prima facie* plausible example of a very odd, but desirable goal. By having the intention to reach the goal, you in effect have the intention of not getting it, because you know that if you have the intention to reach the goal you won't reach it. Conversely, by having the intention not to reach the goal, you have the intention of reaching it. This supposition results in a diagonal intention of achieving φ iff you don't intend to achieve φ. This diagonalized intention appears to yield similar difficulties for the logic of intention. Yet it has nothing to do with direct quotation at least on the face of it; they appear to be properly classified as paradoxes of indirect discourse.

[3]Hans Kamp have investigated a proposal along these lines in Asher & Kamp (1986), Asher(1988), Asher & Kamp (1989).

of the Prior situation to a treatment of various paradoxes of direct discourse. But that approach also has certain drawbacks. A theorem of Asher & Kamp (1989) shows that the full logic of reasoning about belief or knowledge in such a framework is not axiomatizable. Moreover, it remains unclear how to weaken systems in that framework without making certain stipulations on the models that in effect rule out semantic self-reference (see theorem 14 of Asher and Kamp (1989)).

Somewhat surprisingly, a modification of the higher order approach to propositional quantification yields an axiomatizable theory (when we consider general structures) while nevertheless permitting significant possibilities for self-reference. But to get a viable theory of propositional quantification I must give a satisfactory solution to the problem posed by the Prior situation and another general difficulty afflicting higher order theories of propositional quantification. This general difficulty, noticed originally by Russell (1903), is that the simple theory of types is too liberal in what it countenances as propositions and propositional functions, and this forces unintuitive consequences upon the theory. For example, we are forced to say as a truth of logic that there are two propositional functions p and q such that an agent must believe that p = q even though p and q are not coextensive. The core of my proposal is an inductive definition for the propositional quantifiers. This appears to solve both the paradox of the Prior situation and Russell's difficulty.

2. The Language of Higher Order Propositional Quantification and the Intentional, Simple Theory of Types

The first order framework here entails that variables of quantification only occur in argument positions to relational symbols; there are no variables occuring in predicate positions. In particular variables do not occur in 0-place predicate positions-- i.e., in the positions of sentences or formulas. So propositions are quantified over in such a theory, only insofar as they are arguments to properties.

We could, however, quantify also over relations and properties, considering propositions to be 0-place properties. Quantification over predicate positions is the syntactic criterion for a higher order logic. The expressive power of higher order logic is quite attractive when thinking about mathematical theories.[1] There is also evidence in natural language of at least an indirect sort that we do directly quantify over higher order objects, and not just their first order correlates that some have assumed to be the denotions of sentential and verbal nominals. But I won't go into that here.

[1] When we think of a theory like standard set theory or arithmetic we think of a certain canonical structure. We find the Lowenheim Skolem Tarski theorems surprising, even paradoxical when applied to theories of these structures (as we think of them naively) Higher order logic can describe these structures up to isomorphism, and the Lowenheim Skolem Tarski theorems don't hold for higher order theories. For a very good defense of the view that second order logic underlies mathematical practice see Shapiro (1985).

This train of thought leads to a different theory of propositional quantification, the one that Thomason and Prior had in mind.[1] Syntactically, propositional variables and constants are 0-place property variables and constants. The language of propositional quantification, L_2, is thus a second order language. However, I shall consider a natural extension, L_ω, the language of the theory of simple types.[2] Formulas are constructed in the usual manner from the truth functional connectives and quantifiers. L_ω is a language containing individual and temporal constants and variables for all finite types formed from the basic primitive types-- P (the set of propositions), E (the set of individuals) and T (the set of truth values {0, 1}). Formulas are defined for each type using λ-abstraction and functional application. So for instance, if ζ is a formula of type τ and x is a variable of type τ', then $\lambda x\zeta$ is a formula of type $\tau \rightarrow \tau'$, and if ψ is of type $\tau \rightarrow \tau'$ and β is of type τ, then $\psi(\beta)$ is of type τ'.

L_ω has extensional and intentional versions of the connectives and quantifiers. $\forall$, $\exists$, $\&$, $\vee$, $\rightarrow$, $\neg$ will be the truth functional operators and quantifiers, while Π, Σ, $\cap$, $\cup$, $\Rightarrow$ and $\sim$ will be the intensional correlates. Extensional identity, =, also has an intentional correlate, $\approx$. I shall also assume that in the language there is also a function constant $^\vee$ from propositions to their truth values as in Thomason (1980) (manuscript). Note that $^\vee p$ is not considered to be a proposition!

We insure a homomorphism between extensional and intensional correlates if we take the following as axioms:

(HOM)

for all p, q:	$^\vee[p \cap q] = {}^\vee p \,\&\, {}^\vee q$	$^\vee[p \cup q] = {}^\vee p \vee q$	$^\vee[p \Rightarrow q] = {}^\vee p \rightarrow {}^\vee q$
for all ζ:	$^\vee[\Pi x^\tau \zeta] = \forall x^{\tau\vee} \zeta$	$^\vee[\Sigma x^\tau \zeta] = \exists x^{\tau\vee} \zeta$	
for all p:	$^\vee[\sim p] = \neg {}^\vee p$		
for all t, t':	$^\vee[t \approx t'] = {}^\vee[t = t']$		

To get a complete freedom in choosing one's intentional logic for the attitudes, it is better to give for each usual extensional quantifier and connective an intentional operator. But for the statement of various truth definitions, it is very tiresome to read recursive clauses for each quantifier and connective; so in what follows I shall illustrate the various definitions by just exploiting the connectives, quantifiers and operators in the first column of the above table. The rest of the cases are always entirely obvious, and the interested reader may easily fill them in.

Once variables range over sentence denotations, it no longer make sense to take these to be truth values a la Frege, Carnap and Montague, if we wish to justice to propositional attitudes and other intensional

[1]There are arguments for getting rid of the types in doing natural language semantics. But I want to sidestep those here, as they usually revolve around a treatment of properties (with one or more argument places!) and this would lead us too far afield here.
[2]It is interesting to note that some difficulties such as those in the last section of the paper arise in full type theory but not simple quantification over propositions and properties in intentional logic. This seems to cast doubt on the equivalence in intentional logic between second order and full type theory. This equivalence is a fact of extensional, higher order logic.

contexts. Rather, we must take the denotations of sentences to be propositions. A sentence will be true iff the proposition it denotes is true. Thus, (1.a) expresses the proposition,

(3) Πp (believe(mary, p) $\Rightarrow$ p).

(3) is a formula of L_ω; in L_ω 'believe' is a second order predicate of individuals and propositions. By the correspondence rules in (HOM) (7) and hence (1.a) are true just in case,

$\forall p$ ($^\vee$believe(mary, p) $\rightarrow$ $^\vee$p), where p ranges over the domain of propositions.

A *standard intentional model* with times of L_ω consists of a quadruple $<\underline{E}, [\![\,]\!], f, \mathcal{F}>$. $\underline{E}$ is an inductively defined set of domains of various types, with non-empty sets E_0, E_P, E_I and E_T (of individuals, propositions, times and truth values respectively) as the basic types of objects. Other types are constructed from basic types as functions from types to types. In a standard model, if $\tau_1, \ldots \tau_n$ are types, then the set of all objects of type $<\tau_1, \ldots, \tau_n>$, $E(<\tau_1, \ldots, \tau_n>), = \wp(E(\tau_1) \times E(\tau_2) \times \ldots \times E(\tau_n))$. The interpretation of expressions of the other types are the functions constructible from these basic types. I shall also assume that types are closed under functional application.

(FA)

If ν is of type $\tau \rightarrow \tau'$ and ζ of type τ, then $\nu(\zeta) \in E_{\tau'}$

$[\![\,]\!]$ assigns an (intentional) interpretation to each expression of type τ; the interpretation is some element of $E(\tau)$. The interpretation function of an intentional model respects λ abstraction and application in its assignments. That is, we have for any term α of type τ and any term $\lambda x\, \beta$ of type $\tau \rightarrow \tau'$,

(ABS)

$[\![\lambda x \beta]\!]([\![\alpha]\!]) = [\![\beta(x/\alpha]\!]$.

Our theory is intentional; so the objects assigned to predicates of a language by $[\![\,]\!]$ are properties and relations, not sets. Since sets are useful in the truth definition, however, intentional models have a function f that assigns to each object in a type a certain extension. Let [] be the extension of $[\![\,]\!]$ and f to include the assignment of denotations to complex terms of the form $^\vee\varphi$. Then $\mathcal{F}$ is a function from $P \times I$ into $T = \{0, 1\}$ such that:

i. $\mathcal{F}_t(G(a_1, \ldots a_n)) = 1$ iff $<[\![a_1]\!], \ldots, [\![a_n]\!]> \in f([\![G]\!])$

ii. $\mathcal{F}_t(p \cap q) = 1$ iff $\mathcal{F}_t(p) = \mathcal{F}_t(q) = 1$

iii. $\mathcal{F}_t(\sim q) = 1 - \mathcal{F}_t(q)$

iv. $\mathcal{F}_t(\Pi x^\tau \zeta) = 1$ iff $\mathcal{F}_t(\zeta(a)) = 1$ for all objects a of type τ

v. $\mathcal{F}_t(\alpha \approx \beta) = 1$ iff $[\alpha]_t = [\beta]_t$

(similarly for the other operators)

If $[\![\varphi]\!]$ is a proposition, $^\vee\varphi$ is a singular term denoting in $\mathcal{M}$ the truth value of $[\![\varphi]\!]$ in $\mathcal{M}$. It requires a special interpretation. Further these singular terms may combine with truth functional operators and quantifiers,

which will have the usual recursive, semantic clauses. Let us write $[A]_{t,M} = 1$ if A denotes in M truth at t; $[A]_{t,M} = 0$ otherwise.

a. If A is of the form $^{\vee}\varphi$ where $[\![\varphi]\!]_{\mathcal{M}}$ is a proposition, then $[A]_{t,M} = \mathcal{F}(\varphi)$

b. If A is of the form B & C, then $[A]_{t,M} = 1$ iff $[A]_{t,M} = 1$ and $[A]_{t,M} = 1$.

c. If A is of the form $\neg B$, then $[A]_{t,M} = 1$ iff $[A]_{t,M} = 0$.

d. If A is of the form $\forall x^{\tau}\, \zeta$, $[A]_{t,M} = 1$ iff $[\zeta(a/x)]_{t,M} = 1$ for all a of type τ.

e. If A is of the form $^{\vee}[\alpha = \beta]$, $[A]_{t,M} = \mathcal{F}_t(\alpha \approx \beta)$.

f. If A is of the form $at(^{\vee}\varphi, t)$, $[A]_{t,M} = \mathcal{F}_t(\varphi)$.

(similarly for the other operators)

Let T_0 be the theory consisting of the axioms of quantification generalized to higher types, the usual axioms for identity, and the rule of β conversion, closed under the rule modus ponens. Given my definition of intentional models, every intentional model $\mathcal{M}$ for L_ω verifies (HOM) as well as the usual rules of predicate logic and β-conversion in T_0. The models for L_ω impose a structure on P.[1] P is closed under the operations $\cap$, $\sim$, $\cup$ and $\Rightarrow$; Π, Σ must be functions from PF $\rightarrow$ P, where PF is the set of propositional functions $\{f \mid f: E \cup P \rightarrow P\}$. I will take P to be an algebra whose atoms are given by the atomic sentences of L_ω.

Let's now formulate the intentional paradoxes or paradoxes of indirect discourse within this theory. I'll assume some standard addition of constants for times and set of times in the models for $L\omega$. The proposition denoted by (2) is easily expressed in L_ω, and it is true just in case (4) holds.

(4) $(\forall p\, (^{\vee}B(prior, p, t_0) \rightarrow \neg at(^{\vee}p\, t_0) \vee \forall p\, (^{\vee}B(tarski, p, t_1) \rightarrow \neg at(^{\vee}p, t_0))) \,\&$

$(\exists p\, (^{\vee}B(tarski, p, t_0) \,\&\, at(^{\vee}p, t_0)) \vee \exists p\, (^{\vee}B(prior, p, t_1) \,\&\, at(^{\vee}p, t_0)))$

Prior's informal argument now can be stated as follows:

Proposition 1: There is no intentional model for L_ω $\mathcal{M}$ such that Prior thinks (4) at t_0 in $\mathcal{M}$, Tarski thinks that snow is white at t_1 in $\mathcal{M}$ and 'snow is white' is true at t_1 in $\mathcal{M}$.

The paradox of the Prior situation differs from the semantic paradoxes like the Liar and paradoxes of application and comprehension like Russell's predicative paradox. There is no question of inconsistency in the theory T_0 or HOM; and the simple intentional theory of types is after all a highly restricted framework (in comparison, for instance, to ZF). Nevertheless, Prior's thought experiment yields entirely unsatisfactory results.

3. A Semi-Inductive Definition of Propositional Quantification

[1] A couple of facts about $^{\vee}$ are immediate once we realize it is a function constant from propositions into truth values. First of all, $^{\vee}$ does not iterate; so $^{\vee}\,^{\vee}\varphi$ isn't well-defined. Thus any identity statement like $p = {}^{\vee}[\sim p]$ is false in every model! So if we formalize the Liar as such an identity, the Liar is just necessarily false. We might also symbolize the Liar as $^{\vee}p \leftrightarrow \neg\,^{\vee}p$. But this sentence too is false in every model; it is a simple contradiction. Thus, the liar does not pose any problems in this higher order logic. Higher order logic says that the liar is false in every model. Note also that the strong liar, which says that the Liar is false, is logically true in this theory!

The reason why this theory of propositional quantification gets into difficulties is not hard to discover, if we contrast the higher order theory of propositions with the more familiar first order theory of propositional quantification. As the translations for (1.a) and (2) in higher order logic make evident, the truth predicate has disappeared into the theory of propositional quantification. The higher order theory of quantification (as Ramsey and Prior might naturally have suggested) yields a "pro-sentential theory of truth," on which the truth predicate in English is just an anaphor, or perhaps even more simply a dummy or redundant predicate needed because of the limitations of natural language syntax. The theory of quantification has in effect swallowed up the truth predicate. To fix the sort of difficulties that Priorean thought experiments like (2) give rise to, then, the natural suggestion is to do for quantification what Kripke and Gupta-Herzberger have done for predicates like truth. Just as truth is defined inductively or semi-inductively mirroring the restrictions of the Tarskian hierarchy, so too is quantification to be similarly bounded by types until the construction is finished. My proposal complicates the connection between propositions and their truth values in intentional models by using either semi-inductive or inductive definitions for the domains of quantification.

Let me make the suggestion a bit more precise by looking at the semi-inductive case first. Let $\mathcal{M}_0$ be a standard intentional model for type-theory. I distinguish a subset of $P_{\mathcal{M}}$, P_0, which contains just those propositions not containing propositional variables. We now define a *revision sequence* of models $\mathcal{M}_{QH}{}^{\alpha}$ as follows. Let $\mathcal{M}_{QH}{}^{\alpha} = \langle \underline{E}, [\![\,]\!], f, \mathcal{F}^{\alpha} \rangle$. We now define a recursion for $\mathcal{F}^{\alpha}$ on the ordinals. $\mathcal{F}^0 = \mathcal{F}\restriction P_0 \cup (P\text{-}P_0 \times \{0\})$. All the definitions for $\mathcal{F}$ and the assignment of truth values to terms of the form $^{\vee}\varphi$ largely the same as before with the exception of the quantified clauses:

i. $\mathcal{F}_t{}^{\alpha}(G(a_1, \ldots a_n)) = 1$ iff $\langle [\![a_1]\!], \ldots, [\![a_n]\!] \rangle \in f([\![G]\!])$

ii. $\mathcal{F}_t{}^{\alpha}(p \cap q) = 1$ iff $\mathcal{F}_t{}^{\alpha}(p) = \mathcal{F}_t{}^{\alpha}(q) = 1$

iii. $\mathcal{F}_t{}^{\alpha}(\sim q) = 1 - \mathcal{F}_t{}^{\alpha}(q)$

iv. $\mathcal{F}_t{}^{\alpha+1}(\Pi x^{\tau}\, \zeta) = 1$ iff $\mathcal{F}_t{}^{\alpha}(\zeta(a)) = 1$ for all a of type $\tau \neq P$.

v. $\mathcal{F}_t{}^{\alpha}(\alpha \approx \beta) = 1$ iff $[\alpha]_{\mathcal{M}^{\alpha}} = [\beta]_{\mathcal{M}^{\alpha}}$

vi. If A is of the form $^{\vee}\varphi$ where $[\![\varphi]\!]_{\mathcal{M}}$ is a proposition, then $[A]_{\mathcal{M}^{\alpha}} = \mathcal{F}_t{}^{\alpha}(\varphi)$

vii. If A is of the form B & C, then $[A]_{t,\mathcal{M}^{\alpha}} = 1$ iff $[B]_{t,\mathcal{M}^{\alpha}} = 1$ and $[C]_{t,\mathcal{M}^{\alpha}} = 1$.

viii. If A is of the form $\neg B$, then $[A]_{t,\mathcal{M}^{\alpha}} = 1$ iff $[B]_{t,\mathcal{M}^{\alpha}} = 0$.

ix. If A is of the form $\forall x^{\tau}\, \zeta$, $[A]_{t,\mathcal{M}^{\alpha}} = 1$ iff $[\zeta(a^{\tau}/x)]\delta_{t,} = 1$ for all $a^{\tau}\ \tau \neq P$.

x. If A is of the form $^{\vee}[\alpha = \beta]$, $[A]_{t,\mathcal{M}^{\alpha}} = \mathcal{F}_t{}^{\alpha}(\alpha \approx \beta)$.

(similarly for the other operators and non-propositional quantifiers)

The clauses for the propositional quantifiers must be defined relative to previous models in the sequence. We need a pair of clauses for successor and limit ordinal cases.

xi.a. $\mathcal{F}_t^{\alpha+1}(\Pi x^P \zeta) = 1$ iff $\mathcal{F}_t^{\alpha}(\zeta(t^P)) = 1$ for all t^P.

xii.a If A is of the form $\forall x^P \zeta$, $[A]_{t, \mathcal{M}\alpha+1} = 1$ iff $[\zeta(t^P/x)]_{t, \mathcal{M}^\alpha} = 1$ for all t^P.

xi.b. $\mathcal{F}_t^{\lambda}(\Pi x^P \zeta) = 1$ iff $\exists\beta\, \forall\alpha\, (\beta \leq \alpha < \lambda \rightarrow \mathcal{F}_t^{\alpha}(\Pi x^P \varphi) = 1)$.

xii.b If A is of the form $\forall x^P \zeta$, $[A]_{t, \mathcal{M}\lambda} = 1$ iff $\exists\beta\, \forall\alpha\, (\beta \leq \alpha < \lambda \rightarrow [A]_{t, \mathcal{M}\alpha} = 1$.
(similarly for Σx^P)

The first stage of our model revision procedure now may have a quantificational incoherence in there. For instance, a quantificational proposition of the form $\pi x^P \varphi$ will be false in $\mathcal{M}^0$ even though all it's instances may be true. But this incoherence is erased once the revision procedure gets started. We can still show that every model $\mathcal{M}^\alpha{}_{QH}$ in the revision sequence defined verifies (HOM).

Our model revision procedure now yields eventually a *higher order semistable* model, as all sentences with a string of propositional quantifiers of a given depth that will stabilize eventually do so. $\mathcal{M}^\delta$ is a *higher order semistable* model just in case δ is a perfect stabilization ordinal for $\mathcal{M}$ with respect to the revision sequence above and $\mathcal{F}$.[1] Let $\mathcal{M}^\gamma$ be such a model. Prior's belief, (4), is false at $\mathcal{M}^\gamma$, if Tarski's belief is true. Moreover, the truth of Tarski's belief, if it is a simple proposition, does not depend upon Prior's thinking (4) or not thinking (4). So far so good. But a rather surprising result is in store for us:

Proposition 2: There is no semi-inductive model such that such that (i) Prior thinks (4) at t_0 in $\mathcal{M}$ and nothing else, (ii) Tarski thinks that snow is white at t_1 in $\mathcal{M}$ and nothing else, (iii) 'snow is white' is true at t_1 in $\mathcal{M}$, and (iv) $\mathcal{M}$ is a model of T_0.

The proof proceeds by an examination of cases. We observe that on such a theory (4) also has a 2 cycle interpretation. Any $\mathcal{M}^0$ cannot be a model of T_0, because the T_0 theorem $\varphi(c^P) \rightarrow \exists x^P \varphi(x^P)$ is false at $\mathcal{M}^0$, where c^P is a propositional term. Successor states $\mathcal{M}^{\gamma+1}$ either fail to verify $\forall x^P\, \psi(x^P) \rightarrow \psi(c^P/x^P)$, where ψ is either the subformula

$$^{\vee}B(\text{prior}, p, t_0) \rightarrow \neg at(^{\vee}p\, t_0)$$

or

$$^{\vee}B(\text{tarski}, p, t_1) \rightarrow \neg at(^{\vee}p, t_0)$$

of (4); or they share the following difficulty with limit stages $\mathcal{M}^\lambda$. $\mathcal{M}^\lambda \models (4)$ iff $\mathcal{M}^\lambda \models \forall x^P\, (^{\vee}B(a, x^P, t_0) \rightarrow \neg^{\vee} x^P)$ iff $\exists\beta\, \forall\gamma\, (\beta \leq \gamma < \lambda \rightarrow \mathcal{M}^\gamma \models \forall x^P\, (^{\vee}B(a, x^P, t_0) \rightarrow \neg^{\vee} x^P)$. So $\mathcal{M}^\lambda \models \neg(4)$. But then by ordinary quantificational logic, $\mathcal{M}^\lambda \models \neg(4)$ iff $\mathcal{M}^\lambda \models \exists x^P\, (^{\vee}B(a, x^P, t_0)\ \&\ ^{\vee} x^P)$. But by the constraint (i), $\mathcal{M}^\lambda \models \neg(4)$ iff $\mathcal{M}^\lambda \models (4)$.

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[1] Call γ a *perfect stabilization ordinal for* $\mathcal{M}$ with respect to the revision scheme and $\mathcal{F}$ just in case every proposition that comes to have a stable truth value assignment from $\mathcal{F}$ in the revision scheme does so before or at γ and further $\mathcal{F}^\gamma(\varphi) = 1$ in iff $\exists\beta \leq \gamma\, \mathcal{F}^\alpha(\varphi) = 1$ for all $\alpha \geq \beta$.

4. An Inductive Definition of Propositional Quantification

A more satisfactory construction is available with an inductive definition like the one used by Kripke (1975). Let $\mathcal{M}$ be any standard intentional model for L_ω satisfying (FA) and (ABS). Recall that the distinguished subset of $P_{\mathcal{M}}$, P_0, contains just those propositions not containing propositional variables. An inductive revision sequence is defined by setting $\mathcal{F}^0 = \mathcal{F}\restriction P_0$ and the *base partial model* $\mathcal{M}_{QK}{}^0 = \langle \underline{E}, [\![]\!], f, \mathcal{F}^0 \rangle$, $\mathcal{M}_{QK}{}^\alpha = \langle \underline{E}, [\![]\!], f, \mathcal{F}^\alpha \rangle$, and then requiring the following constraint on $\mathcal{F}$ (which I call the *partial model constraint* PMC):

(PMC)

1. $\mathcal{F}^\alpha$ and $[]_{\mathcal{M}}{}^\alpha$ are closed under the usual semantical rules for a strong Kleene interpretation of the truth functional connectives and non-propositional quantifiers.
2. All $\mathcal{M}_{QK}{}^\alpha$ verify identity statements of the form $\beta = \beta$, where β is any term. Otherwise,
   $\mathcal{M}_{QK}{}^\alpha \models \beta = \beta'$ iff $[\beta] = [\beta']$ and both $[\beta]$ and $[\beta']$ are defined in $\mathcal{M}_{QK}{}^\alpha$
   $\mathcal{M}_{QK}{}^\alpha \quad \beta = \beta'$ iff $[\beta] \neq [\beta']$ and both $[\beta]$ and $[\beta']$ are defined in $\mathcal{M}_{QK}{}^\alpha$
3. For propositional quantifiers (Again I illustrate only for Πx^P, the case for Σx^P is entirely analogous)
   A. With regard to the successor case:
      i.a. $\mathcal{F}^{\alpha+1}(\Pi x^P\, \zeta) = 1$ if $\mathcal{F}^\alpha(\, \zeta(t^P)) = 1$ for all t^P
      i.b. $\mathcal{F}^{\alpha+1}(\Pi x^P\, \zeta) = 0$ if $\mathcal{F}^\alpha(\, \zeta(t^P)) = 0$ for some t^P.
      i.c. $\mathcal{F}^{\alpha+1}(\Pi x^P\, \zeta)$ undefined otherwise.
      ii. If A is of the form $\forall x^P\, \zeta$,
         a. $[A]_{\mathcal{M}\alpha+1} = 1$ if $[\zeta(t^P/x)]_{\mathcal{M}}{}^\alpha = 1$ for all t^P.
         b. $[A]_{\mathcal{M}\alpha+1} = 0$ if $[\zeta(t^P/x)]_{\mathcal{M}}{}^\alpha = 1$ for some t^P
         c. $[A]_{\mathcal{M}\alpha+1}$ = undefined otherwise.
   B. The limit case may defined quite simply.
      a. $\mathcal{F}^\lambda = \bigcup_{\beta<\lambda} \mathcal{F}^\beta$
      b. $[]_{\mathcal{M}\lambda} = \bigcup_{\beta<\lambda} []_{\mathcal{M}\beta}$

The QK sequence of models builds up inductively the values of the partial function $\mathcal{F}_\alpha$ and the extensional definition [] for each α. In $\mathcal{M}_{QK}{}^0$ no propositionally quantified statements are given truth values. $\mathcal{F}\restriction P_0$, however, does assign every atom in the propositional algebra a truth value. After the first application of the inductive definition $\mathcal{M}_{QK}{}^1$ now verifies many propositions that quantify over propositions-- e.g. $\exists p\, p$. But notice that (4) will not get a value in $\mathcal{M}_{QK}{}^1$. In fact (4) will not get a value throughout the QK sequence. I will call the models in the QK sequence *standard partial models* for L_ω.

Standard partial models are not models of (HOM) as it stands. But they are models for a closely related theory. We must make two changes to (HOM). First, we must define correspondences for each pair of intensional and extensional connectives, and second we must replace the identities in (HOM) with the corresponding rule equivalences (e.g, replace $^{\vee}[p \cap q] = {}^{\vee}p \,\&\, {}^{\vee}q$ with

$$\frac{{}^{\vee}[p \cap q]}{{}^{\vee}p \,\&\, {}^{\vee}q}$$

and so on. The rule equivalences in (HOM') form a weaker theory from (HOM) to be sure. We only have a partial homomorphism from propositions to truth values respecting the propositional and truth functional connectives and quantifiers. But we can still prove the following with it. Define a L_ω formula φ' in $^{\vee}$ *normal form* such that $^{\vee}$ occurs only in front of atomic formulas. The rules in (HOM') allow us to prove

Proposition 3: Let φ be a formula of L_ω. Then given (HOM'), there is a formula φ' in $^{\vee}$- normal form such that $\varphi \vdash \varphi'$.

Because the QK sequence of models is inductively defined and there is a fixed set of propositions, one can show by the standard argument that the sequence reaches a fixed point. I'll call any $\mathcal{M}_{QK}{}^{\gamma}$ model that is a fixed point of the definition a *standard fixed point* model for L_ω. Let R_1 be the following set of rules (corresponding to the strong Kleene interpretation of the connectives and quantifiers):

1. The usual introduction and elimination rules for $\exists\ \forall\ \&$ and $\vee$.
2. The equivalences

$$\frac{\neg\neg A}{A} \qquad \frac{\neg(A \,\&\, B)}{\neg A \vee \neg B} \qquad \frac{\neg(A \vee B)}{\neg A \,\&\, \neg B}$$

3. The rule $\psi \,\&\, \neg\psi \vdash \varphi$
4. Suppose $\varphi(\psi)$ is a positive context (ψ is a constituent that is not under the scope of any negations or relation symbols in prenex disjunctive form). Then if $\psi_1 \vdash \psi_2$, $\varphi(\psi_1) \vdash \varphi(\psi_2)$.
5. The axioms
   a. $\beta = \beta$
   b. $\lambda x^{\tau} A(\beta) = A(\beta/x)$
6. If α and α' are of type $\tau \rightarrow \tau'$ and β, β' of type τ', then $\alpha = \alpha' \,\&\, \beta = \beta' \vdash \alpha(\beta) = \alpha'(\beta')$
7. $\forall u\ \lambda x\ \varphi(u) = \lambda x\ \varphi'(u) \vdash \lambda x\ \varphi = \lambda x\ \varphi'$.
8. The rules in (HOM')

Proposition 4: if $\mathcal{M}_{QK}{}^{\alpha}$ is a fixed point of the QK sequence, then if $\varphi \vdash \psi$ is a rule of R_1, then if $\mathcal{M}_{QK}{}^{\alpha} \models \varphi$, then $\mathcal{M}_{QK}{}^{\alpha} \models \psi$.

To illustrate, let us take one of the quantifier rules, the universal exploitation rule $\forall x\ \varphi \vdash \varphi\ (t/x)$. Suppose $\mathcal{M}_{QK}{}^{\alpha} \models \forall x\ \varphi$. If x is of other than propositional type, then by the constraints on $\mathcal{F}$ given by the strong

Kleene interpretation of the truth functional connectives and non-propositional quantifiers, $\mathcal{M}_{QK}{}^{\alpha} \models \varphi(t/x)$ for any suitable term t. Now suppose that x is of propositional type. By the construction of the sequence QK $\forall x\varphi$ will be true only if all its instances are verified at some previous stage, if α is a successor or limit ordinal. In either case, since the construction is inductive, this assures that $\mathcal{M}_{QK}{}^{\alpha} \models \varphi(t/x)$.

Let $\vdash_{R_1}$ be the derivation relation defined by the rules in R_1. A standard argument will now prove the soundness of R_1 at fixed points of model sequences QK. Let $\Vdash$ be the consequence relation defined over the class of fixed point models of QK model sequences. Then,

Proposition 5: For a set of sentences Γ, if $\Gamma \vdash_{R_1} \varphi$, then $\Gamma \Vdash \varphi$.

It appears that if we loosen the notion of a standard partial model for L_ω to get *general partial models* L_ω, we may also be able to prove a completness result about R_1. A *general model* for L_ω from Henkin (1950) is a model in which the domains of propositions, truth values, and individuals are as before and where if $\tau_1, \ldots \tau_n$ are types, then the interpretation of a type $< \tau_1, \ldots, \tau_n>$ is a *subset* of $\wp([\![\tau_1]\!] \times [\![\tau_2]\!] \times \ldots \times [\![\tau_n]\!])$; in a standard model $[\![< \tau_1, \ldots, \tau_n>]\!] = \wp([\![\tau_1]\!] \times [\![\tau_2]\!] \times \ldots \times [\![\tau_n]\!])$. Let $\Vdash_G$ be the consequence relation defined over general, fixed point models-- those fixed points that come from the jump operation being applied to general models.

Proposition 6: For a set of sentences Γ, if $\Gamma \Vdash_G \varphi$, then $\Gamma \vdash_{R_1} \varphi$.

An outline of the proof is given in the appendix. Proposition 6 establishes a logic for partial fixed point models of propositional quantification, a logic which I'll call *partial, simple theory of types* (SPT). But it does so by using general models. If we define first and order logic by means of the model theoretic properties of their *standard* models rather than by their syntax, the use of general models for SPT essentially convert higher order logic to first order logic. But in this SPT is no different from the standard simple theory of types (ST); with respect to standard partial intentional models SPT is sound just as with respect to standard intentional models, (ST) is sound. By the completeness proof, logical consequence for SPT relative to the class of general partial models is Σ_1 definable, just as (ST) is Σ_1 definable relative to general models; with respect to standard models, consequence in SPT (and in ST) is not axiomatizable.[1]

5. Russell's Problem with the Theory of Types

Thomason's paper discusses another problem for the simple theory of types, mentioned in an appendix to Russell's *Principles of Mathematics*. It motivates Thomason's proposal for dealing with the intentional paradoxes, which uses a free logic for the propositional quantifiers. My proposal solves this difficulty too, though in a manner different from Thomason's proposal.

The difficulty, due originally to Russell (1903), is that the simple theory of types is too liberal in what it countenances as propositions and propositional functions. For example in L_ω the term

[1]The proof of this claim would follow the lines of that given by Van Bentham and Doets (1984).

(5) $\lambda x^p \exists f^{<p,p>} ({}^\vee Ff = x \;\&\; \neg {}^\vee fx)$

denotes a property of propositions,[1] for any given F. Let's call the property of propositions in (5) w. Then assuming ${}^\vee w(Fw) \vee \neg {}^\vee w(Fw)$, we get the following disturbing result.[2]

(6) $\exists f^{<p,p>} \exists g^{<p,p>} ({}^\vee[Ff = Fg] \;\&\; \neg \forall x^p ({}^\vee fx \leftrightarrow {}^\vee gx))$

Since (6) holds for arbitrary F (underlying it is a simple cardinality argument),[3] it holds for the particular definition of F in (7)

(7) $F = \lambda g^{<p,p>} \forall x^p ({}^\vee gx \rightarrow {}^\vee x)$

By the principles of identity (6) and (7) have worrisome consequences for the theory of attitudes formulated within the simple theory of types. One such consequence is (8):

(8) $\exists f^{<p,p>} \exists g^{<p,p>} (\Box \forall x^o \Box ({}^\vee Bel(x, Ff) \leftrightarrow {}^\vee Bel(x, Fg)) \;\&\; \neg \forall x^p ({}^\vee fx \leftrightarrow {}^\vee gx))$

Thomason (1982) points out correctly that by limiting what expressions denote higher order objects in the models and by employing a free logic, one can avoid this difficulty. Thomason's proposal won't work in the partial logic for propositional quantifiers as I have defined it. It is a valid principle of the partial logic SPT that

(9) $\exists p \; {}^\vee[p = \varphi]$

This partial logic for the theory of types is no different from the classical theory of types in this respect. But Thomason's proposal also leads, as he points out, to unintuitive consequences when dealing with the Intentional Paradoxes: it implies among other things that the existence of propositions is a context dependent, speaker relative matter. This collides with our intuitions about propositions. (9) also appears to

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[1]The superscripts in the formulas (5)-(8) are there to make clear the types of variables involved.

[2]The proof is as follows:
Dropping carrots we have
1) $w(Fw) \vee \neg w(Fw)$
Now suppose that
2) $w(Fw)$
and that
3) $\forall g \forall h (F(g) = F(h) \rightarrow \forall y(g(y) \leftrightarrow h(y)))$
Then by 2) and the definition of w,
4) $\exists f(F(f) = F(w) \;\&\; \neg f(F(w)))$
So for some f_0
4) $F(f_0) = F(w) \;\&\; \neg f_0(F(w))$
By (3) and (5),
6) $\forall y(f_0(y) \leftrightarrow w(y))$
By (2) and (6),
7) $f_0(F(w))$
which is a contradiction. So now suppose
8) $\neg w(Fw)$
By the definition of w again, and 8)
9) $\forall f(F(f) = F(w) \rightarrow f(F(w)))$
So by the laws of identity,
10) $w(F(w))$
Again this is a contradiction. Note that this proof is valid in T_0.

[3]If one thinks of how propositional functions might operate compositionally with propositions in a standard model, cardinality arguments would dictate that the function from a tuple consisting of a propositional function and its arguments to propositions could not be 1-1; (6) then is simply a special case of a much more general argument.

be a needed principle in the analysis of propositional anaphora in natural language. Even though propositions are paradoxical or non-sensical, we may refer to them anaphorically. Imagine the following dialogue:

(10) Cretan: Everything I say is false.

Socrates: I don't believe that.

According to the free logic proposal, the Cretan did not manage to express a proposition in the circumstance in which the first sentence of (10) is the only sentence he manages to utter. But then it appears that Socrates doesn't manage to have a belief-- or express a belief-- about what the Cretan said. The analysis of anaphora in (10) is a semantic mystery, unless we assume there is some proposition the Cretan expresses.

Russell's argument culminating with (6) is not valid in SPT for the simple reason that it relies on the excluded middle. So that motivation for introducing type-free logic for higher order quantification dissolves. It's also not clear, however, that (8) is such a bizarre consequence for a theory of simple types to countenance. The real difficulty hinges on what one takes to be the criterion of identity for types. Our models say little about what identity of types should amount to. One could have criteria of type identity such that $\psi(\beta) = \psi(\beta')$ but $\beta \neq \beta'$.[1] This goes against a certain natural criterion of identity for intentional objects that one might call a *structural* criterion of identity for types (SIT):

(SIT) Let β and β' be of type τ and let ψ, ψ' be of type $\tau \rightarrow \tau'$. Then $\psi(\beta) = \psi(\beta')$ implies $\psi = \psi'$ & $\beta = \beta'$.

(SIT) together with the principle of indiscernibility of identicals contradicts (6). Thus (SIT) + the principle of indiscernibility of identicals is inconsistent with the simple theory of types (ST). There are at least trivial models of (SPT), in which (SIT) + the principle of indiscernibility of identicals are never refuted and are verified in the trivial cases of where $\alpha(\beta) = \alpha(\beta)$ (which must be true according to the constraints on $\mathcal{F}$ in models for SPT).

Russell's problem suggests a comparison of PT with Russell's own solution-- the Ramified Theory of Types (RT). In all versions of (RT), there is a function from propositions to ω that recursively assigns orders. In some versions,[2] it is defined as follows:

Ord: $E_P \rightarrow \omega$ such that:

If $\varphi \in P_0$ (a designated subset of E_P), then $Ord(\varphi) = 1$.

If $\varphi = \alpha = \beta$ then $Ord(\varphi) = Max\{Ord(\alpha), Ord(\beta)\} + 1$.

If $\varphi = \neg\psi$, then $Ord(\varphi) \leq Ord(\psi)$.

If * is a boolean two place connective and $\varphi = \alpha * \beta$, then $Ord(\varphi) \leq Max\{Ord(\alpha), Ord(\beta)\}$.

[1] Aczel (1989) warns that the application relation should not be taken to be structure creating for such reasons. That is, he wants to deny that $\alpha(\beta) = \alpha'(\beta') \rightarrow \alpha = \alpha'$ & $\beta = \beta'$, our principle (SIT).

[2] I folow Thomason (1989) and Church (1976) here.

If Q is a quantifier and $\varphi = Q\delta\psi$, then $Ord(\varphi) = Max\{Ord(\delta)$ for $\delta \in Dom(Q)$ in $\psi\} + 1$, where if φ is of the form $Q\delta(\zeta, \theta)$, $Dom(Q)$ in ψ is just the set of objects satisfying ζ. $Dom(Q)$ in ψ is just $E_{type(\delta)}$ otherwise.

We have in effect constructed models for certain versions of RT. But the orders of our theory are given by the semantics; they are artefacts of the models, and not part of the syntax and formation rules of the language of PT (this is different for RT). For formulas of PT, then, Ord must be a partial function, because there are many propositions in our setup that cannot be assigned an order-- Prior's proposition for instance. The definition of order above then suggests the following correlation between order and stages of revision in our model theoretic framework.

Proposition 7: Suppose φ is a proposition for which Ord is defined. Then $\mathcal{M}^n \models {}^\vee\varphi^*$ iff $Ord(\varphi) \leq n$, where ${}^\vee\varphi^*$ is ${}^\vee\alpha = {}^\vee\beta$, if φ is $\alpha = \beta$ and ${}^\vee\varphi^* = {}^\vee\varphi$ otherwise.

The proof of proposition 7 is by induction on n.

To sum up then, there appear to be two solutions to the paradoxes of indirect discourse. One familiar route uses a first order theory of quantification and a truth predicate. The other uses higher order logic, in particular the intentional version presupposed by Russellians and spelled out in Thomason (1980.b) and a Pro-sentential theory of truth. By giving an inductive definition of propositional quantification, we avoid the difficulties associated with other solutions to the intentional paradoxes concerning in higher order logic. The partiality of SPT is located within what truth values propositions take on, not, as in Thomason's proposal, the existence of propositions. Surprisingly, only an inductive definition not a semi-inductive definition will do the trick.

The pro-sentential theory of truth incorporated into higher order propositional quantification appears to mitigate Liar-like paradoxes. We have no restrictions on the logic of predicates of propositions. This is responsible for the completeness proof for SPT. No such completeness proof is available to the first order theory in general.[1] Somewhat surprisingly, the system with the higher order syntax-- partial type theory-- thus turns out to have a more tractable notion of validity than that of the first order theory of propositions on which a truth predicate is used. The set of valid sentences in all metastable models or semi-stable models,[2] for instance, is clearly not r.e for interesting classes of models.

I have said little so far about logics of knowledge and belief in this theory. But it seems we may accomodate within this framework most reasonable logics for knowledge and belief in a relatively straightforward way . Now suppose our semantics for attitude predicates is such that for every agent we assign a *belief state*, a collection of propositions. We may subject this state to various closure conditions. For instance we may require of every belief state S that if the proposition $p \in S$, then $B(p) \in S$. We may

[1]For a case where we can prove completeness, see theorem 14 of Asher and Kamp (1989).
[2]For definitions of these model classes, See Asher and Kamp (1986), Asher and Kamp (1989).

thus encode by means of these closure conditions the usual doxastic reasoning principles and validate rules which correspond, say, to the logic presented in Thomason (1980.a).[1] SPT, however, permits a variety of logics for the attitudes which go far beyond what a possible worlds framework yields. The reason for this is simple. If our semantics for attitudes ascribes to an agent a set of propositions, then we may choose from a variety of closure conditions that cannot be expressed in a possible worlds semantics. In particular we may assume very weak closure conditions-- such as those detailed in Asher (1986). Nothing forces us in SPT to require a closure condition on S that exploits logical equivalence. In SPT it is consistent to assume that two propositions may be necessarily even logically equivalent without being identical. So SPT does not validate B(p) and $\vdash p \leftrightarrow q \Rightarrow B(q)$, an inference form that is valid in most possible worlds semantics of attitudes. Thus, SPT easily avoids problems about logical omniscience.

The guarantees we have in L_ω that eliminate possibilities of expressing a proposition corresponding to the Liar sentence (see footnote 20) do not stop us from attempting similar definitions, say, with a knowledge or belief predicate. Let 'B' be a two place predicate representing the relation of belief between individuals and propositions. Suppose we stipulate in the semantics for a propositional constant c, $[\![c]\!] = \neg B(a, c)$. Alternatively, it seems as though we could imagine a theory in our language in which:

(11) $p = \sim B(a, p)$

By our constraints on $\mathcal{F}$ it follows that in every model in which (11) is true,

(12) $^{\vee}p \nvdash \neg {}^{\vee}B(a, p)$

Now suppose our semantics for attitude predicates is such that for every agent we assign a *belief state*, a collection of propositions which is subject to closure conditions validating the logic presented in Thomason (1980). We can still have such identities between propositions as in (11). But p will never get assigned a truth value, and so it will be undetermined whether a believes p. We must be careful not introduce any "essentially ungrounded" propositions with such predicates as knowledge and belief into our domain; if we do so completeness will vanish (we can no longer construct the models) and the higher order theory of propositions becomes an uninteresting variant of the first order theory, as far as I can tell at present.[2] Well, we can't have everything. SPT is one option for treating propositional quantification that has some attractions. It yields a theory of propositions with a tractable logic and that countenances some self-referential propositions, while avoiding problems of logical omniscience with the semantics of the attitudes.

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[1]Here would be the relevant closure principles for the S4 logic of Thomason (1980.a):

$p \rightarrow q, p \in S \Rightarrow q \in S$
$p \in S \Rightarrow Bp \in S$
$p \in S$ and $p \vdash q \Rightarrow q \in S$.
$Bp \in S \Rightarrow p \in S$.

[2]Thus for instance we cannot introduce an expression relation between sentences and propositions. See Parsons (1974), Asher & Kamp (1986) for a discussion. Nevertheless, one can still in this setup have beliefs about mathematics.

Appendix: Proof of Proposition 6

The outline of the proof relies on an adaptation of the Henkin method to partial models proposed by Kamp (1984). What I shall do is show that if not $\Gamma \vdash_{R_1} \varphi$ then there is a partial model that verifies Γ but does not verify φ. So suppose not $\Gamma \vdash \varphi$. Define φ to be a positive formula just in case all negation signs in φ occur only on atomic formulae. We may show that for every φ there is a positive φ' that is R-equivalent to it (i.e. $\varphi \vdash \varphi'$).,

Using an enumeration of all positive formulae of L_ω, we build up two maximal sets Ω and Σ from Γ and $\{\varphi\}$ respectively as follows. I assume that infinitely many constants of each type do not occur in the enumeration of the positive formulae.

1. $\Omega_0 = \Gamma$; $\Sigma_0 = \{\varphi\}$
2.a. if not $(\Omega_n \cup \{\psi_{n+1}\} \vdash \Sigma_n)$ and ψ_{n+1} is not existential, then $\Omega_{n+1} = \Omega_n \cup \{\psi_{n+1}\}$; $\Sigma_{n+1} = \Sigma_n$
   b. if not $(\Omega_n \cup \{\psi_{n+1}\} \vdash \Sigma_n)$ and $\psi_{n+1} = \exists v\zeta$, then $\Omega_{n+1} = \Omega_n \cup \{\psi_{n+1}, \zeta(c_j/v)\}$ where c_j is the first individual constant not appearing in $\Omega_n \cup \Sigma_n \cup \{\psi_{n+1}\}$; $\Sigma_{n+1} = \Sigma_n$
   c. if $\Omega_n \cup \{\psi_{n+1}\} \vdash \Sigma_n$ and ψ_{n+1} is not universal, then $\Omega_{n+1} = \Omega_n$; $\Sigma_{n+1} = \Sigma_n \cup \{\psi_{n+1}\}$
   d. if $\Omega_n \cup \{\psi_{n+1}\} \vdash \Sigma_n$ and $\psi_{n+1} = \forall v\zeta$, then $\Omega_{n+1} = \Omega_n$; $\Sigma_{n+1} = \Sigma_n \cup \{\psi_{n+1}, \zeta(c_j/v)\}$ where c_j is the first individual constant not appearing in $\Omega_n \cup \Sigma_n \cup \{\psi_{n+1}\}$
3. $\Omega = \bigcup_{n \in \omega} \Omega_n$; $\Sigma = \bigcup_{n \in \omega} \Sigma_n$

The next step is to show

Lemma 6.1: not $\Omega \vdash \Sigma$

This is proved by an induction on Ω_n and Σ_n.

I now construct a base partial intentional model $\mathcal{M} = \langle \underline{E}, [\![\,]\!], f, \mathcal{F}^0 \rangle$ from these sets. First I inductively define the type structure $\underline{E}$. Let $E_{0\mathcal{M}} = \{[c^0]_\Omega$: c is an individual constant occurring in $\Omega \cup \Sigma\}$, where$[c]_\Omega = \{d : \Omega \vdash d = c\}$, and let $E_{p\,\mathcal{M}} = \{[\psi]_\Omega$: ψ is a sentence occuring in $\Omega \cup \Sigma\}$. I define E_T as the set of truth values using the sentences and their negates in Ω. $\top = \{\psi: \psi \in \Omega\}$; $\bot = \{\psi: \neg\psi \in \Omega\}$. Now assume that E_τ and $E_{\tau'}$ are already defined as equivalence classes $[\alpha]_\Omega$ and $[\beta]_\Omega$ respectively. We define $E_{\tau \to \tau'}$ to be $\{[\zeta]_\Omega$: ζ is of the form $\lambda x^\tau \gamma$ and γ occurs in $\Omega \cup \Sigma\}$. We further define for each $[\zeta]_\Omega$ in $E_{\tau \to \tau'}$ to be a function such that $[\zeta]_\Omega([\alpha]_\Omega) = [\gamma(x^\tau/\alpha)]_\Omega$ for any element $[\alpha]_\Omega$ of E_τ. We can easily check this definition by exploiting (6) and (7) of R_1 and noting that for any $\alpha_1, \alpha_2 \in [\alpha]_\Omega$ and $\lambda x\gamma_1, \lambda x\gamma_2$ in $[\zeta]_\Omega$, $\lambda x\gamma_1(\alpha_1) = \lambda x\gamma_2(\alpha_2)$. Further, if $[\zeta_1]_\Omega = [\zeta_2]_\Omega$, it follows that ζ_1 and ζ_2 agree on all arguments. So $\forall u$ $\lambda x\gamma_1(x)(u) = \lambda x\gamma_2(x)(u)$. By (7) we may conclude $\zeta_1 = \zeta_2$. The set of all types $\underline{E}$ for $\mathcal{M}$ are those constructed from the basic types by this procedure, and it obeys (FA).

The second step in defining the model is to specify the interpretation function. Define $[\![\,]\!]$ as follows. If φ is a term of type τ, then $[\![\varphi]\!] = [\varphi]_\Omega \in E_\tau$. Because of my definition of the type structure and because of (5.b), $[\![\,]\!]$ obeys (ABS).

The third step is to specify extensions for intentional objects. Define f such that for $[\beta]_\Omega \in E_{\langle \tau_1, \ldots, \tau_n \rangle}$ $f(x) = \{\langle [\alpha_1]_\Omega, \ldots, [\alpha_n]_\Omega \rangle$: $\beta(\alpha_1, \ldots \alpha_n) \in \Omega\}$.

The final step is to assign truth values to propositions. Define $\mathcal{F}^0_{\mathcal{M}}$ to be a function from E_{P_0} to E_T such that: if φ is of the form $R(\beta_1, \ldots \beta_n)$, then $\mathcal{F}_{\mathcal{M}}(R(\beta_1, \ldots \beta_n)) = 1$ iff $R(\beta_1, \ldots \beta_n) \in \Omega$ and $\mathcal{F}_{\mathcal{M}}(R(\beta_1, \ldots \beta_n)) = 0$ iff $R(\beta_1, \ldots \beta_n) \in \Sigma$.. Given my definition of f, $\mathcal{F}^0$ is correctly defined.

Now extend $\mathcal{F}^0_{\mathcal{M}}$ to a partial function $\mathcal{F}^\alpha$ from E_P to E_T using the inductive revision procedure defined in (PMC). Let $\mathcal{M}^\alpha = \langle E, P, [\![\,]\!], f, \mathcal{F}^\alpha \rangle$ be the fixed point of that revision process.

Lemma 6.2: $\mathcal{M}^\alpha$ is a partial model that verifies all of Ω and fails to verify Σ. Hence $\mathcal{M}^\alpha$ verifies Γ and fails to verify φ.

We prove this by induction on the complexity of $\vartheta \in \Omega \cup \Sigma$. Suppose that ϑ is atomic of the form $R(\beta_1, \ldots \beta_n)$ The construction of $\mathcal{F}^0$ insures that $\mathcal{M} \models \vartheta$ if $\vartheta \in \Omega$ and not $\mathcal{M} \models \vartheta$ if $\vartheta \in \Sigma$. Suppose $\vartheta = \neg\psi$ and that $\vartheta \in \Omega$. By the construction of Ω, $\psi \notin \Omega$ and ψ must be atomic. But then $\psi \in \Sigma$ and so again by the definition of $\mathcal{F}^0$, $\mathcal{M} \models \psi$ and so $\mathcal{M} \models \vartheta$. An entirely parallel argument holds if $\vartheta \in \Sigma$. The truth functional cases and ordinary quantificational cases are straightforward. Suppose $\vartheta = \lambda x\alpha(\beta) \in \Omega$. By (5.b) in R_1, α'

$= \alpha(\beta/x) \in \Omega$, and by the inductive hypothesis $\mathcal{M} \models \alpha'$ if $\alpha' \in \Omega$. So $\mathcal{M} \models \alpha'$. Since $\mathcal{M}$ obeys (FA) and (ABS) as seen above, $\mathcal{M} \models \vartheta$. A similar argument holds for the case $\vartheta \in \Sigma$.
The only non-straightforward step involves quantified statements of the form $\exists p\psi$ and $\forall p\ \psi$ where p is a propositional quantifier. Let $\vartheta = \exists p\psi$ and suppose $\vartheta \in \Omega$. By the construction of Ω, if $\exists p\psi \in \Omega$, then $\psi(c^p_j/p) \in \Omega$. By the inductive hypothesis $\mathcal{M}^\alpha \models \psi(c^p_j/p)$ and so $\mathcal{M}^\alpha \models \exists p\psi$, since $\mathcal{M}^\alpha$ is a fixed point. Now suppose that $\vartheta \in \Sigma$. $\vartheta \in \Sigma$ only if it implies φ or is itself φ. So by the construction procedure of Σ and Ω, every instance $\psi(c^p_j/p)$ of ψ must be in Σ, since $\psi(c^p_j/p) \vdash \exists p\psi$. But $\mathcal{M}^\alpha \models \exists p\psi$ iff for some proposition c^p_j, $\mathcal{M}^\alpha \models \psi(c^p_j/p)$, since $\mathcal{M}^\alpha$ is a fixed point. Then by the inductive hypothesis it is not the case that $\mathcal{M}^\alpha \models \psi(c^p_j/p)$ for any instance $\psi(c^p_j/p)$ of ψ, and so not $\mathcal{M}^\alpha \models \exists p\psi$. The arguments where $\vartheta = \forall p\psi$ are analogous to those for the existential case. Suppose $\vartheta \in \Omega$. We must show $\mathcal{M}^\alpha \models \forall p\ \psi$. By the construction procedure and the fact that $\forall p\psi \vdash \psi(c^p_j/p)$, every instance $\psi(c^p_j/p) \in \Omega$, and by the inductive hypothesis $\mathcal{M}^\alpha \models \psi(c^p_j/p)$. So $\mathcal{M}^\alpha \models \forall p\ \psi$. Now assume that $\varphi \in \Sigma$. By the construction of Σ, an instance $\psi(c^p_j/p) \in \Sigma$. By the inductive hypothesis then, not $\mathcal{M}^\alpha \models \psi(c^p_j/p)$. But this suffices to show that t is not the case that $\mathcal{M}^\alpha \models \vartheta$.

From Lemma 6.2, it now immediately follows that we have a model of Ω that fails to verify φ, and the proof of proposition 6 is done.

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